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LOCAL COHOMOLOGY AND SUPPORT FOR TRIANGULATED CATEGORIES

BY DAVE BENSON*, SRIKANTH B. IYENGAR AND HENNING KRAUSE

To Lucho Avramov, on his 60th birthday.

ABSTRACT. – We propose a new method for defining a notion of support for objects in any compactly generated triangulated category admitting small coproducts. This approach is based on a construction of local cohomology functors on triangulated categories, with respect to a central ring of operators. Special cases are, for example, the theory for commutative noetherian rings due to Foxby and Neeman, the theory of Avramov and Buchweitz for complete intersection local rings, and varieties for representations of finite groups according to Benson, Carlson, and Rickard. We give explicit examples of objects, the triangulated support and cohomological support of which differ. In the case of group representations, this allows us to correct and establish a conjecture of Benson.

RÉSUMÉ. – Nous proposons une façon nouvelle de définir une notion de support pour les objets d'une catégorie avec petits coproduits, engendrée par des objets compacts. Cette approche est basée sur une construction des foncteurs de cohomologie locale sur les catégories triangulées relativement à un anneau central d'opérateurs. Comme cas particuliers, on retrouve la théorie pour les anneaux noethériens de Foxby et Neeman, la théorie d'Avramov et Buchweitz pour les anneaux locaux d'intersection complète, ou les variétés pour les représentations des groupes finis selon Benson, Carlson et Rickard. Nous donnons des exemples explicites d'objets dont le support triangulé et le support cohomologique diffèrent. Dans le cas des représentations des groupes, ceci nous permet de corriger et d'établir une conjecture de Benson.

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1. Introduction

> Herr K. sagte einmal: “Der Denkende benützt kein Licht zuviel, kein Stück Brot zuviel, keinen Gedanken zuviel.”
> BERTOLT BRECHT, Geschichten von Herrn Keuner

The notion of support is a fundamental concept which provides a geometric approach for studying various algebraic structures. The prototype for this has been Quillen’s [49] description of the algebraic variety corresponding to the cohomology ring of a finite group, based on which Carlson [22] introduced *support* varieties for modular representations. This has made it possible to apply methods of algebraic geometry to obtain representation theoretic information. Their work has inspired the development of analogous theories in various contexts, notably modules over commutative complete intersection rings, and over cocommutative Hopf algebras.

In this article we propose a new method for defining a notion of support for objects in any compactly generated *triangulated category* admitting small coproducts. The foundation of our approach is a construction of *local cohomology* functors on triangulated categories, with respect to a central ring of operators; this is inspired by work of Grothendieck [32]. Suitably specialized our approach recovers, for example, the support theory of Foxby [27] and Neeman [47] for commutative noetherian rings, the theory of Avramov and Buchweitz for complete intersection local rings [3, 6], and varieties for representations of finite groups, according to Benson, Carlson, and Rickard [16]. It is surprising how little is needed to develop a satisfactory theory of support. To explain this, let us sketch the main results of this paper.

Let T be a triangulated category that admits small coproducts and is compactly generated. In the introduction, for ease of exposition, we assume T is generated by a single compact object C_0. Let $Z(\mathsf{T})$ denote the graded center of T. The notion of support presented here depends on the choice of a graded-commutative noetherian ring R and a homomorphism of rings

$$R \longrightarrow Z(\mathsf{T})\,.$$

We may view R as a ring of cohomology operators on T. For each object X in T its cohomology

$$H^*(X) = \operatorname{Hom}^*_{\mathsf{T}}(C_0, X) = \coprod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(C_0, \Sigma^n X)$$

has a structure of a graded module over $Z(\mathsf{T})$ and hence over R. We let $\operatorname{Spec} R$ denote the set of graded prime ideals of R. The *specialization closure* of a subset $\mathcal{U} \subseteq \operatorname{Spec} R$ is the subset

$$\operatorname{cl}\mathcal{U} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \text{there exists } \mathfrak{q} \in \mathcal{U} \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}\,.$$

This is the smallest specialization closed subset containing $\mathcal{U}$.

One of the main results of this work is an axiomatic characterization of support:

THEOREM 1. – *There exists a unique assignment sending each object X in T to a subset $\operatorname{supp}_R X$ of $\operatorname{Spec} R$ such that the following properties hold:*

(1) Cohomology: *For each object X in T one has*

$$\mathrm{cl}(\operatorname{supp}_R X) = \mathrm{cl}(\operatorname{supp}_R H^*(X))\,.$$

(2) Orthogonality: *For objects X and Y in T, one has that*

$$\mathrm{cl}(\operatorname{supp}_R X) \cap \operatorname{supp}_R Y = \varnothing \quad \textit{implies} \quad \operatorname{Hom}_{\mathsf{T}}(X, Y) = 0\,.$$

(3) Exactness: *For every exact triangle $W \to X \to Y \to$ in T, one has*

$$\operatorname{supp}_R X \subseteq \operatorname{supp}_R W \cup \operatorname{supp}_R Y\,.$$

(4) Separation: *For any specialization closed subset $\mathcal{V}$ of* $\operatorname{Spec} R$ *and any object X in T, there exists an exact triangle $X' \to X \to X'' \to$ in T such that*

$$\operatorname{supp}_R X' \subseteq \mathcal{V} \quad \textit{and} \quad \operatorname{supp}_R X'' \subseteq \operatorname{Spec} R \setminus \mathcal{V}\,.$$

Implicit in (1) is a comparison of the *triangulated support* $\operatorname{supp}_R X$ and the *cohomological support* $\operatorname{supp}_R H^*(X)$. This was part of the initial motivation for this work. We prove also that if the cohomology $H^*(X)$ is finitely generated as a module over R, then $\operatorname{supp}_R X = \operatorname{supp}_R H^*(X)$. Without such finiteness assumption however, triangulated and cohomological support can differ; see Sections 9 and 10 for explicit examples.

It is thus interesting that the triangulated support of an object X can be yet detected by cohomology. Only, one has to compute cohomology with respect to *each compact object*. This is made precise in the next result, where, for a graded R-module M, we write $\min_R M$ for the set of minimal primes in its support.

THEOREM 2. – *For each object X in T, one has an equality:*

$$\operatorname{supp}_R X = \bigcup_{C \text{ compact}} \min_R \operatorname{Hom}^*_{\mathsf{T}}(C, X)\,.$$

In particular, $\operatorname{supp}_R X = \varnothing$ *if and only if* $X = 0$.

Beyond proving Theorems 1 and 2, we develop systematically a theory of supports in order to make it a viable tool. For example, in Section 7, we establish the following result of Krull-Remak-Schmidt type.

THEOREM 3. – *Each object X in T admits a unique decomposition $X = \coprod_{i \in I} X_i$ with $X_i \neq 0$ such that the subsets* $\mathrm{cl}(\operatorname{supp}_R X_i)$ *are connected and pairwise disjoint.*

Here is a direct corollary: If X is an indecomposable object in T, then $\operatorname{supp}_R X$ is a connected subset of $\operatorname{Spec} R$. This generalizes and unifies various connectedness results in the literature, starting with a celebrated theorem of Carlson, which states that the variety of an indecomposable group representation is connected [23].

As stated before, the basis for this work is a construction of local cohomology functors on T. Given a specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} R$, we establish the existence of (co)localization functors $\Gamma_{\mathcal{V}}$ and $L_{\mathcal{V}}$ on T, such that for each X in T there is a natural exact triangle

$$\Gamma_{\mathcal{V}} X \longrightarrow X \longrightarrow L_{\mathcal{V}} X \longrightarrow$$

in T. We view $\Gamma_{\mathcal{V}}$ as the local cohomology functor with respect to $\mathcal{V}$. One justification for this is the following result:

$$\operatorname{supp}_R \Gamma_{\mathcal{V}} X = \mathcal{V} \cap \operatorname{supp}_R X \quad \text{and} \quad \operatorname{supp}_R L_{\mathcal{V}} X = \big(\operatorname{Spec} R \setminus \mathcal{V}\big) \cap \operatorname{supp}_R X\,.$$

A major focus of this work are properties of the functors $\Gamma_{\mathcal{V}}$ and $L_{\mathcal{V}}$ for a general triangulated category T; the results on support are derived from them. These occupy Sections 4–7 in this article; the first three prepare the ground for them, and for later sections. The remaining sections are devoted to various specific contexts, and are intended to demonstrate the range and applicability of the methods introduced here. We stress that hitherto many of the results established were known only in special cases; Theorem 3 is such an example. Others, for instance, Theorems 1 and 2, are new in all contexts relevant to this work.

In Section 8 we consider the case where the triangulated category T admits a symmetric tensor product. The notion of support then obtained is shown to coincide with the one introduced by Hovey, Palmieri, and Strickland [36].

Section 9 is devoted to the case where T is the derived category of a commutative noetherian ring A, and $R = A \to Z(\mathsf{T})$ is the canonical morphism. We prove that for each specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} A$ and complex X of A-modules the cohomology of $\Gamma_{\mathcal{V}} X$ is classical local cohomology, introduced by Grothendieck [32].

The case of modules for finite groups is studied in Section 10, where we prove that support as defined here coincides with one of Benson, Carlson, and Rickard [16]. Even though this case has been studied extensively in the literature, our work does provide interesting new information. For instance, using Theorem 2, we describe an explicit way of computing the support of a module in terms of its cohomological supports. This, in spirit, settles Conjecture 10.7.1 of [14] that the support of a module equals the cohomological support; we provide an example that shows that the conjecture itself is false.

The final Section 11 is devoted to complete intersection local rings. We recover the theory of Avramov and Buchweitz for support varieties of finitely generated modules [3, 6]. A salient feature of our approach is that it gives a theory of local cohomology with respect to rings of cohomology operators.

This article has influenced some of our subsequent work on this topic: in [9], Avramov and Iyengar address the problem of realizing modules over arbitrary associative rings with prescribed cohomological support; in [41], Krause studies the classification of thick subcategories of modules over commutative noetherian rings. Lastly, the techniques introduced here play a pivotal role in our recent work on a classification theorem for the localizing subcategories of the stable module category of a finite group; see [17].

2. Support for modules

In this section R denotes a $\mathbb{Z}$-graded-commutative noetherian ring. Thus we have $x \cdot y = (-1)^{|x||y|} y \cdot x$ for each pair of homogeneous elements x, y in R.

Let M and N be graded R-modules. For each integer n, we write $M[n]$ for the graded module with $M[n]^i = M^{i+n}$. We write $\operatorname{Hom}^*_R(M,N)$ for the graded homomorphisms between M and N:

$$\operatorname{Hom}^n_R(M,N) = \operatorname{Hom}_R(M, N[n])\,.$$