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ON NON-BASIC RAPOPORT-ZINK SPACES

BY Elena MANTOVAN

Abstract. – In this paper we study certain moduli spaces of Barsotti-Tate groups constructed by Rapoport and Zink as local analogues of Shimura varieties. More precisely, given an isogeny class of Barsotti-Tate groups with unramified additional structures, we investigate how the associated (non-basic) moduli spaces compare to the (basic) moduli spaces associated with its isoclinic constituents.

This aspect of the geometry of the Rapoport-Zink spaces is closely related to Kottwitz's prediction that their l-adic cohomology groups provide a realization of certain cases of local Langlands correspondences and in particular to the question of whether they contain any supercuspidal representations.

Our results are compatible with this prediction and identify many cases when no supercuspidal representations appear. In those cases, we prove that the l-adic cohomology of the non-basic spaces is equal (in the appropriate sense) to the parabolic induction of the l-adic cohomology of some associated lower-dimensional (and in the most favorable cases basic) Rapoport-Zink spaces. Such an equality was originally conjectured by Harris in [11] (Conjecture 5.2, p. 420).

Résumé. – Dans cet article, on considère certains espaces de Rapoport-Zink non-ramifiés, associés à des groupes p-divisibles non-basiques et on étudie leur géométrie vis-à-vis de celle des espaces de Rapoport-Zink basiques correspondants.

L'origine de ce problème se situe, d'une part, dans la conjecture de Kottwitz concernant la réalisation des correspondances de Langlands locales dans la cohomologie étale l-adique des espaces de Rapoport-Zink et, d'autre part, plus simplement dans la question d'identifier pour lesquels de ces espaces la partie supercuspidale de la cohomologie n'est pas vide.

Nos résultats sont compatibles avec cette conjecture et, dans certains cas particuliers, ils répondent à la dernière question. En particulier, dans ces cas, on établit une formule reliant la cohomologie de ces espaces à l'induction parabolique de celle de certains espaces de Rapoport-Zink de dimension inférieure (et dans les cas plus favorables basiques). Cette formule a été précédemment conjecturée par Harris dans [11] (Conjecture 5.2, p. 420).

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1. Introduction

1.1. – In [31] Rapoport and Zink introduce some local analogues of (PEL) type Shimura varieties, in the category of rigid analytic spaces over a p-adic local field. (PEL) type Shimura varieties arise as moduli spaces of abelian varieties with additional structures, namely endomorphisms, polarizations and level structures. Similarly, the spaces constructed by Rapoport and Zink are moduli spaces of Barsotti-Tate groups with the analogous additional structures.

A conjecture of Langlands predicts that some cases of correspondences between automorphic representations and global Galois representations are realized inside the cohomology of Shimura varieties. Analogously, a conjecture of Kottwitz (which is heuristically compatible with the conjecture of Langlands) predicts that some cases of correspondences between admissible representations of p-adic groups and local Galois representations are realized in the cohomology of the spaces constructed by Rapoport and Zink. (For linear groups over a local field, both the local case of Langlands' conjecture and Kottwitz's conjecture have been proved, respectively in [12], and [8].)

In the global case, the construction of Shimura associates to certain algebraic groups $G/\mathbb{Q}$ (together with a conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$) a projective system of varieties defined over a number field E (called the *reflex field*), whose cohomology groups (regarded as an inductive limit) are naturally representations of the product of two groups: the points of G over the finite adeles of $\mathbb{Q}$, $G(\mathbb{A}_f)$, and the absolute Galois group of E. Conjecturally, these groups realize the correspondence between automorphic representations of G and representations of the global Weil group of E. In the local case, the construction of Rapoport and Zink depends not only on a choice of an algebraic group $G/\mathbb{Q}_p$ (together with a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G$) but also on a further datum b associated with (G, μ). Moreover, the cohomology groups of the Rapoport-Zink spaces (again regarded as an inductive limit) are naturally representations of the product of three groups: the p-adic group $G(\mathbb{Q}_p)$, the Weil group of the local reflex field, W_E, and a second p-adic group J_b depending on the new data, also of the form $\mathrm{J}_b = J_b(\mathbb{Q}_p)$, for an algebraic group $J_b/\mathbb{Q}_p$.

The presence of a third group raises important new questions. If we maintain our focus on the algebraic group G, a first goal is understanding the role played by the data (b, μ), and in particular, whether all admissible pairs would be relevant in a proof of the existence of the local Langlands correspondence for the group G via the study of the cohomology of the Rapoport-Zink spaces. This question amounts to investigate for which pairs (b, μ) the cohomology of the associated Rapoport-Zink spaces contains supercuspidal representations of $G(\mathbb{Q}_p)$.

More completely, one would like to understand the role played by the group J_b. Indeed, Kottwitz's conjecture predicts that the cohomology of the Rapoport-Zink spaces not only realizes (some cases of) the local Langlands correspondence for the group G, but also for the group J_b, for each b. Furthermore, the obvious compatibility between the two correspondences (due to the fact that they are realized inside the same cohomology groups) would be an example of Langlands' functoriality principle (for each b the group J_b is an inner form of a Levi subgroup of G). Equivalently, the cohomology groups of the Rapoport-Zink spaces conjecturally also realize a generalized Jacquet-Langlands correspondence for J_b and G.

Kottwitz's predictions provide a conjectural answer to the first question we raised. In fact, they imply that supercuspidal representations of $G(\mathbb{Q}_p)$ should appear only in the cohomology groups of the Rapoport-Zink spaces associated with pairs (b,μ) for which J_b is an inner form of G (such pairs are called *basic*). More precisely, they predict that, for any admissible pair b, the admissible representations of $G(\mathbb{Q}_p)$ which arise in the cohomology of the associated Rapoport-Zink spaces are parabolically induced from the Levi subgroup of $G(\mathbb{Q}_p)$ which is an inner form of J_b.

Extending Kottwitz's predictions, in [11] (Conjecture 5.2, p. 420) Harris conjectured that the l-adic cohomology of non-basic Rapoport-Zink spaces is equal (in the appropriate Grothendieck group) to the non-normalized parabolic induction of the l-adic cohomology of the corresponding basic spaces, for a specified choice of the associated parabolic subgroup. Such a reduction of the computation of the l-adic cohomology of the Rapoport-Zink spaces associated with a group G to that of the basic cases for G and its Levi subgroups can be viewed as mirroring the results describing the admissible representations of $G(\mathbb{Q}_p)$ in terms of the supercuspidal representations of $G(\mathbb{Q}_p)$ and of its Levi subgroups. In the case of $G = GL_n$ and $\mu = (0,\dots,0,1)$ (i.e. in the case of Drinfeld's modular varieties) Harris' conjecture was already known, due to the work of Boyer ([4]), and played an important role in the proof of the existence of the local Langlands' correspondence for GL_n in [12], and consequently also in [8].

1.2. – Let $G = \mathrm{Res}_{F_0/\mathbb{Q}_p}(G_0)$, for $F_0/\mathbb{Q}_p$ an unramified finite extension and $G_0 = GL_n$ or GSp_{2n}, for some integer $n \geq 1$. The goal of this paper is to investigate the above conjectures for the admissible pairs (b,μ) associated with such a group $G/\mathbb{Q}_p$. In particular, we will establish some instances of a variant of the conjecture of Harris.

Let us fix μ and consider the set $B(G,\mu)$ of all possible b's such that the pair (b,μ) is admissible. This is defined as a subset of the set of σ-conjugacy classes in $G(K)$, for K the maximal unramified extension of $\mathbb{Q}_p$ and σ its Frobenius automorphism. Its definition is originally due to Kottwitz who in [17] and [19] studied the set $B(G)$ of all σ-conjugacy classes in $G(K)$, for G any connected reductive group over $\mathbb{Q}_p$. This set classifies isomorphism classes of F-isocrystals with G-structures over k, for k the residue field of the ring of integers of K. Indeed, each element $b \in G(K)$ defines an exact faithful tensor functor N_b from the category of p-adic representations of G to that of F-isocrystals over k, via $N_b(W,\rho) = (W \otimes K, \rho(b)(\mathrm{id}_W \otimes \sigma))$. It follows from the definition that any such functor N is defined by a unique $b \in G(K)$, and that if b, b' are σ-conjugate in $G(K)$ then the corresponding functors $N_b, N_{b'}$ are isomorphic. For each $b \in G(K)$, the group J_b is by definition the group of automorphisms of the F-isocrystal with G-structures N_b. (Thus if b, b' are σ-conjugate in $G(K)$ then the associated groups $J_b, J_{b'}$ are isomorphic.) In the cases we are interested in, any such functor N is uniquely determined by its value on the natural representation of G. Therefore, in these cases, an F-isocrystal with G-structures (defined as an exact faithful tensor functor) is simply an F-isocrystal (in the classical sense) together with additional structures, namely given endomorphisms and/or a non-degenerate alternating pairing.

We return to these cases, i.e. to G as at the beginning of the section, and denote by V the natural representation of G. To each $b \in G(K)$, we associate the Newton polygon ν_b of the corresponding F-isocrystal with G-structures over k, $N_b(V)$. Then the set $B(G,\mu)$ is realized as a subset of the set of convex polygons with integral break-points and the same end-points (*Newton polygons*), all lying above a fixed convex polygon, also with integral break-points and the same end-points, associated with μ (the *Hodge polygon*). It follows that there is a natural partial order on the set $B(G,\mu)$: for any two elements b, b' in $B(G,\mu)$, we say $b \geq b'$ if all points of ν_b lie below or on $\nu_{b'}$. Under this partial ordering (which is called the *Bruhat ordering*), $B(G,\mu)$ has unique maximal element, which is called *μ-ordinary* (and the corresponding admissible pair *ordinary*), and a unique minimal element corresponding to the basic pair. (A group-theoretic description of the *Newton map* for G any connected reductive group over $\mathbb{Q}_p$ is discussed by Rapoport and Richartz in [30].)

Let b_0 be the μ-ordinary element in $B(G,\mu)$. The following definition is justified by the results in [24] (we report on them in section 6). We say that an element $b \in B(G,\mu)$ (or the corresponding admissible pair (b,μ))) is *of (HN) type* if there is a break-point x of ν_b which lies on ν_{b_0} and the two polygons coincide up to x or from x on. We call such a break-point x of ν_b also of (HN) type. We remark that when G is symplectic the polygon ν_b is symmetric (for any b). Thus, for each break-point x of ν_b there is an associated symmetrical break-point $\hat{x}$. Furthermore, it is an easy observation that x is of (HN) type if and only if $\hat{x}$ is of (HN) type. To each b of (HN) type we attach a Levi subgroup M_b of G as follows. Every break-point $x = (x_1, x_2) \in \mathbb{Z}^2$ of ν_b defines a decomposition of the F-isocrystal $N_b(V) = V^1 \oplus V^2$, for V^1, V^2 the two sub-F-isocrystals of $N_b(V)$ characterized by the properties that V^1 has Newton polygon ν_1 consisting of the first x_1 slopes of ν_b and V^2 has Newton polygon ν_2 consisting of the remaining slopes of ν_b. Then, to any subset S of the set of break-points of ν_b we associate the unique common refinement of the decompositions of $N_b(V)$ corresponding to each $x \in S$. It follows from the definition that any such decomposition is coarser than or equal to the slope decomposition (i.e. the decomposition of $N_b(V)$ into isoclinic factors), which in these notations is the decomposition associated with the set of all break-points of ν_b. In particular, it follows from the analogous statements for the slope decomposition that any such decomposition of V is F_0-linear, for $F_0/\mathbb{Q}_p$ the field extension as at the beginning of the section, and in the case when G is symplectic compatible with the symplectic pairing on V if the set S is symmetrical, i.e. satisfying the condition $x \in S$ if and only if $\hat{x} \in S$. For each $b \in B(G,\mu)$ of (HN) type we define M_b to be the stabilizer in G of the decomposition of V into p-adic vector subspaces, which underlies the decomposition of the F-isocrystal $N_b(V)$ associated with the set of all break-points of ν_b of (HN) type. It follows from the definition that M_b is a Levi subgroup of G. Finally, we also write L_b for the stabilizer in G of the decomposition of V underlying the slope decomposition of $N_b(V)$. Then, L_b is also a Levi subgroup of G and it is an inner form of J_b (see [17], Section 5.2, p. 215; [31], Corollary 1.14, p. 11) . It follows from the definition that for each b of (HN) type $M_b \supseteq L_b$, and $M_b = L_b$ if all the break-points of ν_b are of (HN) type (i.e. when all but possibly one of the sides of ν_b lie on the μ-ordinary polygon ν_{b_0}).

In this paper, we prove that if the admissible pair (b,μ) is of (HN) type then the l-adic cohomology groups of the associated Rapoport-Zink spaces contain no supercuspidal representations. More precisely, we prove, for pairs of (HN) type, a variant of Harris' conjecture which shows that, as representations of $G(\mathbb{Q}_p)$, these cohomology groups are parabolically induced from $M_b(\mathbb{Q}_p)$ to $G(\mathbb{Q}_p)$. In the special cases when $M_b = L_b$, e.g. for $b = b_0$, we prove Harris' conjecture.