ISSN 0012-9593

quatrième série - tome 41 fascicule 5 septembre-octobre 2008

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 41, 2008, p. 743 à 788

EXPLICIT COMPUTATIONS OF ALL FINITE INDEX BIMODULES FOR A FAMILY OF II_1 FACTORS*

BY Stefaan VAES

ABSTRACT. – We study II_1 factors M and N associated with good generalized Bernoulli actions of groups having an infinite almost normal subgroup with the relative property (T). We prove the following rigidity result : every finite index M-N-bimodule (in particular, every isomorphism between M and N) is described by a commensurability of the groups involved and a commensurability of their actions. The fusion algebra of finite index M-M-bimodules is identified with an extended Hecke fusion algebra, providing the first explicit computations of the fusion algebra of a II_1 factor. We obtain in particular explicit examples of II_1 factors with trivial fusion algebra, i.e. only having trivial finite index subfactors.

RÉSUMÉ. – Nous étudions des facteurs M et N de type II_1 associés à de bonnes actions Bernoulli généralisées de groupes Γ et Λ ayant un sous-groupe infini presque-distingué avec la propriété (T) relative. Nous démontrons le résultat de rigidité suivant : chaque M-N-bimodule d'indice fini (en particulier, chaque isomorphisme entre M et N) peut être décrit par une commensurabilité des groupes Γ, Λ et une commensurabilité de leurs actions. L'algèbre de fusion des M-M-bimodules d'indice fini est identifiée avec une algèbre de Hecke étendue, ce qui fournit les premiers calculs explicites de l'algèbre de fusion d'un facteur de type II_1. Nous obtenons en particulier des exemples explicites de facteurs II_1 dont l'algèbre de fusion est triviale, ce qui veut dire que tous leurs sous-facteurs d'indice fini sont triviaux.

Introduction

To every probability measure preserving action $\Gamma \curvearrowright (X, \mu)$ of a countable group, is associated a tracial von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$, through the group measure space construction of Murray and von Neumann [16]. In the passage from group actions to von Neumann algebras, a lot of information gets lost. Indeed, by the celebrated results of Connes, Feldman, Ornstein and Weiss [7, 18], all free ergodic actions of amenable groups (and even

*Partially supported by Research Programme G.0231.07 of the Research Foundation – Flanders (FWO) and the Marie Curie Research Training Network Non-Commutative Geometry MRTN-CT-2006-031962.

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all amenable type II_1 equivalence relations) systematically yield the same von Neumann algebra, called the hyperfinite II_1 factor.

Recently, Sorin Popa, in his breakthrough articles [22, 23], proved a completely opposite rigidity result : for the first time, he was able to provide a family of group actions such that isomorphism of the crossed product von Neumann algebras, implies isomorphism of the groups involved and conjugacy of their actions. More precisely, Popa proves in [23] the following von Neumann strong rigidity theorem : let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic action of an ICC *w-rigid group,* i.e. a group admitting an infinite normal subgroup with the relative property (T), and let $\Lambda \curvearrowright (Y_0, \eta_0)^\Lambda$ be a Bernoulli action of an ICC group Λ. If the corresponding group measure space II_1 factors are isomorphic, then the groups Γ and Λ are isomorphic and their actions conjugate.

The crucial idea of Popa is the *deformation/rigidity principle*. One studies von Neumann algebras that exhibit both a deformation property (e.g. a specific flow of automorphisms, or a sequence of completely positive unital maps tending to the identity) and a rigidity property (e.g. a subalgebra with the relative property (T)). The tension between both properties determines in a sense the position of the rigid part and allows in certain cases to completely unravel the structure of the studied von Neumann algebra. The deformation/rigidity principle has been successfully applied in a lot of articles. Without being complete, we cite [9, 11, 12, 13, 21, 22, 23, 24, 25] and we explain some aspects of these works below. We also refer to [28] for a survey of some of these results.

The deformation/rigidity principle allows in particular to compute *invariants of II_1 factors.* In [21], Popa proved that the group von Neumann algebra $\mathcal{L}(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2)$ has *trivial fundamental group.* This was the first such example, answering a question of Kadison that remained open since 1967. Here, it should be noticed that Connes proved in [5] that the fundamental group of the group von Neumann algebra $\mathcal{L}(\Gamma)$ is countable whenever Γ is a group with property (T) and infinite conjugacy classes (ICC).

In [22, 23], Popa made a thorough study of *Bernoulli actions* $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ and their noncommutative versions, called Connes-Størmer Bernoulli actions. In [22], this leads to the first constructions of II_1 factors with a *prescribed countable subgroup of $\mathbb{R}^*_+$ as fundamental group.* Alternative constructions have been given since then in [11, 13]. In [23], Popa proves the von Neumann strong rigidity theorem stated in the first paragraph. As an application, he gets the following description of the *outer automorphism group* of the associated II_1 factors. Given the Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ of an ICC w-rigid group Γ, the outer automorphism group of the associated II_1 factor is the semidirect product of the group of characters of Γ and the normalizer of Γ inside $\mathrm{Aut}(X, \mu)$. Up to now, the actual computation of this normalizer remains an open problem though.

In [13], the deformation/rigidity principle was applied to study *amalgamated free product II_1 factors.* From the many far-reaching results obtained in [13], we quote the existence theorem of II_1 factors M with $\mathrm{Out}(M)$ a prescribed abelian compact group. In particular, it was shown that the outer automorphism group of a II_1 factor can be trivial, answering a question posed by Connes in 1973. Using the same techniques, it was shown in [9], that there exist II_1 factors with $\mathrm{Out}(M)$ an arbitrary compact group.

Some of the results on Bernoulli actions obtained in [22, 23], were extended by Popa and the author [25], to include *generalized Bernoulli actions* $\Gamma \curvearrowright (X_0, \mu_0)^I$, associated with an action of Γ on a countable set I. As a result, the first *explicit examples* of II_1 factors with *trivial outer automorphism group* were given. It should be noticed that the shift from plain to generalized Bernoulli actions is not only technical in nature : the former are *mixing* and this is extensively used in [22, 23], while the latter are only *weakly mixing*.

In the current article, we study *bimodules (Connes' correspondences) of finite Jones index* between II_1 factors given by generalized Bernoulli actions. Bimodules between von Neumann algebras were studied by Connes (see V.Appendix B of [6]) and Popa [20]. An M-N-bimodule of finite Jones index (see [14]) can be considered as a *commensuration of M and N*, i.e. an isomorphism modulo finite index. Using the Connes tensor product, the set $\mathrm{FAlg}(M)$ of (equivalence classes of) finite index M-M-bimodules carries the structure of a *fusion algebra* and contains the outer automorphism group $\mathrm{Out}(M)$ as group-like elements.

As a natural follow-up of [22, 23, 25], we provide a family of good generalized Bernoulli actions of groups admitting an infinite almost normal subgroup with the relative property (T) and prove the following rigidity property : any finite index bimodule between the associated II_1 factors comes from a commensurability of the groups and a commensurability of the actions. This allows us to get the following results.

- We provide the first *explicit computations of fusion algebras* for a family of II_1 factors. If the II_1 factor M is defined by a good generalized Bernoulli action $\Gamma \curvearrowright (X,\mu) = (X_0,\mu_0)^I$, the fusion algebra $\mathrm{FAlg}(M)$ is identified with the *extended Hecke fusion algebra* $\mathcal{H}_{\mathrm{rep}}(\Gamma < G)$ of the Hecke pair $\Gamma < G$, where G denotes the commensurator of Γ inside the group of permutations $\mathrm{Perm}(I)$. Loosely speaking, the extended Hecke algebra $\mathcal{H}_{\mathrm{rep}}(\Gamma < G)$ is an extension of the usual Hecke algebra $\mathcal{H}(\Gamma < G)$ by the fusion algebra of finite dimensional unitary representations of Γ. In 2.9 below, we give several concrete examples yielding II_1 factors whose fusion algebras are the extended Hecke algebras of Hecke pairs appearing naturally in arithmetic.
- We give the first *explicit examples of II_1 factors M with trivial fusion algebra,* associated with the generalized Bernoulli action $(\mathrm{SL}(2,\mathbb{Q}) \ltimes \mathbb{Q}^2) \curvearrowright (X_0,\mu_0)^{\mathbb{Q}^2}$ and a scalar 2-cocycle. Equivalently, every finite index subfactor $N \subset M$ is trivial, i.e. isomorphic with $1 \otimes N \subset \mathrm{M}_n(\mathbb{C}) \otimes N$. Note that we proved in [29] the existence of such II_1 factors M, using the techniques of [13].
- Compared to [25], we impose less stringent conditions on the generalized Bernoulli actions involved and obtain more general results on outer automorphism groups. We prove that the actions $\mathrm{PSL}(n,\mathbb{Z}) \curvearrowright (X_0,\mu_0)^{\mathrm{P}(\mathbb{Q}^n)}$ for n odd and $n \geq 3$, provide II_1 factors with *trivial outer automorphism group*. In fact, we provide a concrete construction procedure to obtain II_1 factors with a *prescribed countable group* as an outer automorphism group. The case of groups of finite presentation was dealt with in [25].

Acknowledgment. I would like to thank the referee for pointing out a gap in the original proof of Lemma 4.2.

1. Preliminaries and notations

All von Neumann algebras in this article have separable predual and all Hilbert spaces are separable. All *-homomorphisms between von Neumann algebras are implicitly assumed to be normal. Unless the contrary is explicitly stated, all *-homomorphisms are unital, as well as von Neumann subalgebras.

Let M be a von Neumann algebra. One calls H_M a *(right) M-module* if H is a Hilbert space equipped with a weakly continuous right module action of M. If M is a II_1 factor and if we denote by $\mathrm{L}^2(M)$ the Hilbert space obtained by the GNS construction with respect to the unique tracial state of M, every M-module H_M is isomorphic with an M-module of the form $p(\ell^2(\mathbb{N})\otimes \mathrm{L}^2(M))$, for some projection $p \in \mathrm{B}(\ell^2(M))\overline{\otimes} M$. The projection p is uniquely determined up to equivalence of projections and one defines $\dim(H_M) := (\mathrm{Tr}\otimes\tau)(p)$. All this was already known to Murray and von Neumann (Theorem X in [16]).

The *Jones index* of a subfactor $N \subset M$ of a II_1 factor is defined as $[M : N] := \dim(\mathrm{L}^2(M)_N)$, see [14]. If $[M : N] < \infty$, we call $N \subset M$ a *finite index subfactor* or a *finite index inclusion.*

If (M, τ) is a tracial von Neumann algebra with possibly non-trivial center, the dimension $\dim(H_M)$ of a right M-module H_M is defined similarly, but depends on the choice of trace τ. In this article, there will always be an obvious choice of τ, so that we freely use the notation $\dim(H_M)$.

For any von Neumann algebra M, we denote $M^n := \mathrm{M}_n(\mathbb{C}) \otimes M$ and $M^\infty := \mathrm{B}(\ell^2(\mathbb{N}))\overline{\otimes} M$.

Let N and M be von Neumann algebras. An *N-M-bimodule* ${}_N H_M$ is a Hilbert space H equipped with commuting, weakly continuous, left N-module and right M-module actions. An N-M-bimodule ${}_N H_M$ between tracial von Neumann algebras (N, τ_1) and (M, τ_2) is said to be *of finite Jones index* if $\dim({}_N H) < \infty$ and $\dim(H_M) < \infty$. Bimodules between von Neumann algebras were studied by Connes (see V.Appendix B in [6]) who called them *correspondences*, and by Popa [20].

If M is a II_1 factor, $\mathrm{FAlg}(M)$ is defined as the set of equivalence classes of finite index M-M-bimodules. We call $\mathrm{FAlg}(M)$ the *fusion algebra of the II_1 factor M.*

First recall that an *abstract fusion algebra* $\mathcal{A}$ is a free $\mathbb{N}$-module $\mathbb{N}[\mathcal{G}]$ equipped with the following additional structure :

- an associative and distributive product operation, and a multiplicative unit element $e \in \mathcal{G}$,
- an additive, anti-multiplicative, involutive map $x \mapsto \overline{x}$, called *conjugation*,

satisfying Frobenius reciprocity: defining the numbers $m(x, y; z) \in \mathbb{N}$ for $x, y, z \in \mathcal{G}$ through the formula
$$xy = \sum_z m(x, y; z) z$$
one has $m(x, y; z) = m(\overline{x}, z; y) = m(z, \overline{y}; x)$ for all $x, y, z \in \mathcal{G}$. The base $\mathcal{G}$ of the fusion algebra $\mathcal{A}$ is canonically determined : these are exactly the non-zero elements of $\mathcal{A}$ that cannot be expressed as the sum of two non-zero elements. The elements of $\mathcal{G}$ are called the *irreducible elements* of the fusion algebra $\mathcal{A}$.

If M is a II_1 factor, the fusion algebra structure on $\mathrm{FAlg}(M)$ is given by the direct sum and the *Connes tensor product* of M-M-bimodules. Whenever $\psi : M \to pM^n p$ is a finite index inclusion, define the M-M-bimodule $H(\psi)$ on the Hilbert space $p(\mathrm{M}_{n,1}(\mathbb{C}) \otimes \mathrm{L}^2(M))$ with left and right module action given by $a \cdot \xi = \psi(a)\xi$ and $\xi \cdot a = \xi a$. Every element of $\mathrm{FAlg}(M)$