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QUANTIFICATION POUR LES PAIRES SYMÉTRIQUES ET DIAGRAMMES DE KONTSEVICH

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RÉSUMÉ. – In this article we use the expansion for biquantization described in [7] for the case of symmetric spaces. We introduce a function of two variables $E(X,Y)$ for any symmetric pairs. This function has an expansion in terms of Kontsevich's diagrams. We recover most of the known results though in a more systematic way by using some elementary properties of this E function. We prove that Cattaneo and Felder's star product coincides with Rouvière's for any symmetric pairs. We generalize some of Lichnerowicz's results for the commutativity of the algebra of invariant differential operators and solve a long standing problem posed by M. Duflo for the expression of invariant differential operators on any symmetric spaces in exponential coordinates. We describe the Harish-Chandra homomorphism in the case of symmetric spaces by using all these constructions. We develop a new method to construct characters for algebras of invariant differential operators. We apply these methods in the case of σ-stable polarizations.

ABSTRACT. – Dans cet article nous appliquons les méthodes de bi-quantification décrites dans [7] au cas des espaces symétriques. Nous introduisons une fonction $E(X,Y)$, définie pour toutes paires symétriques, en termes de graphes de Kontsevich. Les propriétés de cette fonction permettent de démontrer de manière unifiée des résultats importants dans le cas des paires symétriques résolubles ou quadratiques. Nous montrons que le star-produit décrit dans [7] coïncide, pour toute paire symétrique, avec celui de Rouvière. On généralise un résultat de Lichnerowicz sur la commutativité d'algèbres d'opérateurs différentiels invariants et on résout un problème de M. Duflo sur l'écriture, en coordonnées exponentielles, des opérateurs différentiels invariants sur tout espace symétrique. On décrit l'homomorphisme d'Harish-Chandra en termes de graphes de Kontsevich. On développe une théorie nouvelle pour construire des caractères des algèbres d'opérateurs différentiels invariants. On applique ces méthodes dans le cas des polarisations σ-stables.

Introduction

Let $(\mathfrak{g},\sigma)$ be a symmetric pair: viz., $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{R}$, while σ is an involution and a Lie algebra automorphism of $\mathfrak{g}$. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the decomposition relative to σ, with $\mathfrak{k}$ and $\mathfrak{p}$ the $+1$- and -1-eigenspaces, respectively (*Cartan's decomposition*).

The Poincaré–Birkhoff–Witt (PBW) theorem ensures the following decomposition of the universal enveloping algebra $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = U(\mathfrak{g}) \cdot \mathfrak{k} \oplus \beta(S(\mathfrak{p}))$$

with β the symmetrization map from $S(\mathfrak{g})$ into $U(\mathfrak{g})$. [(1)] One can then identify, as vector spaces, the symmetric algebra $S(\mathfrak{p})$ of $\mathfrak{p}$ with $U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{k}$ via β. Though $U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{k}$ is not an algebra in general, its $\mathfrak{k}$-invariant subspace $\big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{k}\big)^{\mathfrak{k}}$ is an algebra.

This is of fundamental importance as it is the algebra of invariant differential operators on the symmetric space G/K associated to the symmetric pair $(\mathfrak{g}, \sigma)$, and as such it occurs in harmonic analysis on symmetric spaces in a crucial way.

This algebra is commutative [21, 12], and the PBW theorem ensures that

$$\Big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{k}\Big)^{\mathfrak{k}} \quad \text{and} \quad S(\mathfrak{p})^{\mathfrak{k}}$$

are isomorphic as vector spaces. Observe that $S(\mathfrak{p})^{\mathfrak{k}}$ is just the associated graded of $\Big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{k}\Big)^{\mathfrak{k}}$.

It is conjectured [32] that these two algebras are isomorphic *as algebras*, what the second author has called the *polynomial conjecture*. This is a generalization for symmetric pairs of Duflo's isomorphism [11] between the center $U(\mathfrak{g})^{\mathfrak{g}}$ of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ and the invariant subalgebra $S(\mathfrak{g})^{\mathfrak{g}}$ of its symmetric algebra.

Recall that for every homogeneous space G/H the algebra of G-invariant differential operators may be identified, thanks to a result of Koornwinder [20], with the algebra $\big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ (where $\mathfrak{h}$ denotes the Lie algebra of H). The latter is not commutative in general. Its associated graded is then a Poisson subalgebra of $\big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ with a natural Poisson structure.

If $\mathfrak{h}$ admits a complement $\mathfrak{q}$ which is invariant under the adjoint action of $\mathfrak{h}$, then an easy consequence of PBW is that $\big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ and $\big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ are still isomorphic as vector spaces. It is not known whether this holds in general, for there is no natural map from $\big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ to $U(\mathfrak{g})$ (or a quotient thereof).

It has been conjectured anyway by M. Duflo [13] that the center of $\big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ and the Poisson center of $\big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}}$ are always isomorphic as algebras.

Little is known in such generality. In case $\mathfrak{g}$ is a nilpotent Lie algebra, appreciable advances have been achieved in the last few years by Corwin–Greenleaf [10], Fujiwara–Lion–Magneron–Mehdi [14], Baklouti–Fujiwara [5] and Baklouti–Ludwig [6]. In case where G and H are reductive groups, F. Knop [18] gives a satisfying and remarkable answer to the conjecture. In case where H is compact and $G = H \rhd\!< N$ is the semidirect product of H with a Heisenberg group N, Rybnikov [30] makes use of Knop's result to prove Duflo's conjecture in that case.

In this paper we propose a novel approach to these questions based on Kontsevich's construction [19] and its extension to the case of coisotropic submanifolds by Cattaneo and

(1) We have $\beta(X_1 \dots X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \dots X_{\sigma(n)}$.

Felder [7], [8]. We only treat the problem of symmetric pairs here, but we think that our methods have a wider scope, namely in the nilpotent homogeneous case (2).

One may regard the present work as a link between the methods of deformation quantization and the orbit method in Lie theory.

Plan of the paper

In Section 1 we recall Kontsevich's construction for the deformation quantization of Poisson manifolds and its extension by Cattaneo and Felder to the case of coisotropic submanifolds. We discuss in details the compatibility in cohomology. This section should be useful for the Lie algebra experts who are not familiar with the deformation quantization constructions.

Next we study the dependency of this construction on the choice of a complement in the linear case. We show that the reduction spaces are isomorphic and describe the isomorphism (Proposition 2 and Theorem 1) which is an element of the gauge group obtained by solving a differential equation.

In Section 2 we describe the graphs appearing in the linear case and present three fundamental examples of reduction spaces occurring in Lie theory: symmetric pairs (Proposition 4), Iwasawa's decompositions (Proposition 5), and polarizations (Proposition 6). These examples are new and show that this novel quantization methods are well-suited for Lie theory.

Recall that for symmetric pairs F. Rouvière [26, 27, 29], following Kashiwara and Vergne [17], introduced a mysterious function $e(X,Y)$ defined for $X, Y \in \mathfrak{p}$. Up to conjugation this function computes the star product

$$P \underset{\mathrm{Rou}}{\sharp} Q = \beta^{-1}\Big(\beta(P)\cdot\beta(Q) \quad \text{modulo} \quad U(\mathfrak{g})\cdot\mathfrak{k}\Big)$$

for P, Q in $S(\mathfrak{p})^{\mathfrak{k}}$.

In Section 3 we define a function $E(X,Y)$ for $X, Y \in \mathfrak{p}$ in terms of graphs. This function will behave as Rouvière's function $e(X,Y)$. The comparison of these two functions is a key point of this paper. The function $E(X,Y)$ expresses the Cattaneo–Felder star product $\underset{\mathrm{CF}}{\star}$ in the case of symmetric pairs. By inspection of the graphs appearing in its construction, we obtain a symmetry property (Lemma 11) together with some additional properties in the solvable case (Proposition 8), in the case of Alekseev–Meinrenken symmetric pairs (Proposition 10) as well as in the case of very symmetric quadratic pairs (Proposition 11). In all these cases we show that the function E is identically equal to 1. These elementary but remarkable properties yield new and uniform proofs of results obtained by Rouvière in the solvable case (Theorem 2, Section 3.4 and Proposition 9) and generalize a theorem by Alekseev and Meinrenken (Theorem 3) in the (anti-invariant) quadratic case.

In Section 4 we show that the Cattaneo–Felder and the Rouvière star products coincide (Theorem 4). This is a new result. From this we deduce (Theorem 5) the commutativity for all $z \in \mathbb{R}$ of the algebras $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{k}^{z\mathrm{tr}_k}\Big)^{\mathfrak{k}}$ of invariant differential operators on z-densities, thus generalizing a result by Duflo [12] and Lichnerowicz [21].

(2) One of our students is working on this case.

Another problem posed by M. Duflo in [13] is solved in Section 4: the expression in exponential coordinates of invariant differential operators (Theorem 6). Our solution is given in terms of Kontsevich's graphs.

We define at the end of Section 4 a deformation along the axis of the Campbell-Hausdorff formula for symmetric pairs in the spirit of the Kashiwara-Vergne conjecture. We proved (Theorem 7 and Proposition 13) that this deformation in the case of quadratic Lie algebras, considered as very symmetric quadratic pairs, implies the Kashiwara–Vergne conjecture. We conjecture that in the case of Lie algebras considered as symmetric pairs our E function is equal identically to 1. This conjecture would solve the Kashiwara-Vergne conjecture.

In Section 5 we consider the Harish-Chandra homomorphism for symmetric pairs. Actually there are two natural choices for a complement of $\mathfrak{k}^{\perp}$, one by Cartan's decomposition and the other by Iwasawa's. We show that these two choices together lead to the Harish-Chandra homomorphism for general symmetric pairs. In this language the Harish-Chandra homomorphism consists of the restriction to the little symmetric pair in Iwasawa's decomposition. The former decomposition and the intertwinement then yield a formula for the Harish-Chandra homomorphism in terms of graphs. It follows from this expression that the Harish-Chandra homomorphism is invariant under the action of the generalized Weyl group. We hope that this formula will allow a resolution of the polynomial conjecture for symmetric pairs.

Finally, in Section 6 we apply the principles of bi-quantization [7] to the case of triplets $f+\mathfrak{b}^{\perp}, \mathfrak{g}^{*}, \mathfrak{k}^{\perp}$, where $\mathfrak{b}$ is a polarization for $f \in \mathfrak{k}^{\perp}$. These constructions produce characters for the algebras of invariant differential operators (Proposition 19). It is a novel method that we hope will be promising in other situations as well (3).

In the case of polarizations in normal position we show, by a homotopy on the coefficients (which makes use of an 8 color form), that the characters are independent of the choice of polarization (Proposition 21). Thus we recover some classical results of the orbit method for Lie algebras.

It follows that for symmetric pairs admitting σ-stable polarizations Rouvière's isomorphism computes the characters of the orbit method (Theorem 8).

One can regard these new methods as a replacement for the orbit method.

Introduction

Soit $(\mathfrak{g}, \sigma)$ une paire symétrique, c'est-à-dire $\mathfrak{g}$ est une algèbre de Lie (quelconque) de dimension finie sur $\mathbb{R}$ et σ est une involution qui est un automorphisme d'algèbres de Lie. On note alors $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ la décomposition relative à σ, où $\mathfrak{k}$ désigne l'espace propre associé à la valeur propre $+1$ et $\mathfrak{p}$ l'espace propre associé à la valeur propre -1. Cette décomposition est aussi appelée *décomposition de Cartan*.

Le théorème de Poincaré-Birkhoff-Witt (PBW) assure la décomposition de l'algèbre enveloppante $U(\mathfrak{g})$:

$$U(\mathfrak{g})=U(\mathfrak{g})\cdot\mathfrak{k}\oplus\beta(S(\mathfrak{p})),$$

(3) These methods may be applied in certain cases of homogeneous spaces.