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NONLINEAR COMPRESSIBLE VORTEX SHEETS IN TWO SPACE DIMENSIONS

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Abstract. – We consider supersonic compressible vortex sheets for the isentropic Euler equations of gas dynamics in two space dimensions. The problem is a free boundary nonlinear hyperbolic problem with two main difficulties: the free boundary is characteristic, and the so-called Lopatinskii condition holds only in a weak sense, which yields losses of derivatives. Nevertheless, we prove the local existence of such piecewise smooth solutions to the Euler equations. Since the a priori estimates for the linearized equations exhibit a loss of regularity, our existence result is proved by using a suitable modification of the Nash-Moser iteration scheme. We also show how a similar analysis yields the existence of weakly stable shock waves in isentropic gas dynamics, and the existence of weakly stable liquid/vapor phase transitions.

Résumé. – Nous construisons des nappes de tourbillon supersoniques pour les équations d'Euler compressibles isentropiques en deux dimensions d'espace. Il s'agit d'un problème non-linéaire hyperbolique à frontière libre présentant deux difficultés principales : la frontière libre est caractéristique et la condition dite de Lopatinskii n'est satisfaite que dans un sens faible, ce qui induit des estimations à perte. Néanmoins nous montrons l'existence de telles solutions régulières par morceaux des équations d'Euler en utilisant un schéma itératif de type Nash-Moser palliant les pertes de régularité. Notre analyse s'étend au cas de discontinuités non-caractéristiques et faiblement stables comme certaines ondes de choc pour les équations d'Euler ou les transitions de phase liquide- vapeur.

1. Introduction

The Cauchy problem for the compressible Euler equations in several space dimensions is a major challenge in the domain of hyperbolic conservation laws. The (local in time) existence of smooth solutions away from vacuum follows from a general Theorem by Kato [20], while the existence of smooth solutions with vacuum is proved by Chemin in [8]. Due to the finite time blow-up of smooth solutions, see [38], it is natural to look for weak solutions. The construction of (local in time) piecewise smooth solutions is a preliminary step. The first breakthrough in this direction is the existence of one multidimensional uniformly stable shock wave, that was obtained by Majda [24, 23], see also [6] for a different approach. The existence of two uniformly stable shock waves was shown by Métivier [27]. Then the existence of multidimensional rarefaction waves was obtained by Alinhac [1]. More recently, Francheteau and Métivier [14] have studied the asymptotic behavior of multidimensional shock waves when the strength of the shock tends to zero. The limit of such weak shock waves are sonic waves, whose existence has been proved by Métivier [28]. All these works are based on an appropriate iterative scheme (either a standard Picard iteration or a Nash-Moser iteration), that is proved to converge thanks to a tame estimate on the linearized equations. In this work, we show the existence of contact discontinuities in two space dimensions for the isentropic Euler equations. A similar analysis could be done for the nonisentropic Euler equations, since the stability properties of contact discontinuities for the isentropic Euler equations, and for the nonisentropic Euler equations are quite similar (1).

Let us recall briefly the important features of Majda's work on shock waves. The existence result [23] was obtained under a *uniform stability* assumption, that ensures a good a priori estimate for the linearized equations. By "good" a priori estimate, we mean an estimate where there is no loss of regularity from the source terms to the solution. However, this uniform stability condition is not satisfied by all shock waves in gas dynamics (2). Furthermore, this uniform stability condition (or more precisely the analogue of this condition for characteristic discontinuities), is never satisfied by contact discontinuities in two or three space dimensions, see e.g. [30, 13] or [37, page 222]. As a matter of fact, in three space dimensions, every contact discontinuity is violently unstable (this violent instability is the analogue of the Kelvin-Helmholtz instability for incompressible fluids), while in two space dimensions, a large jump of the tangential velocity makes the contact discontinuity weakly stable. A precise study of this *weak stability* has been performed in [12], where we have shown that for such *weakly stable* contact discontinuities, the linearized equations satisfy an a priori estimate with a loss of one derivative. In this case, we cannot hope to prove the existence of solutions to the nonlinear problem by means of a Picard iteration. In this paper, we shall show that a suitable Nash-Moser iteration converges towards a contact discontinuity solution to the Euler equations.

At the end of the paper, we give two other situations where our analysis applies. More precisely, we can apply the same type of iteration scheme to show the existence of weakly stable shock waves in two or three space dimensions, and the existence of liquid/vapor phase transitions in two or three space dimensions. Roughly speaking, our work shows that the weak Lopatinskii condition, that is known to be sufficient for linear well-posedness [10], is also sufficient for nonlinear well-posedness (even when the verification of the weak Lopatinskii condition is submitted to nonlinear constraints). However, we prefer not to give the proof of such an abstract result, and we shall focus on the problem of contact discontinuities for the

(1) We refer the reader to [30, 13, 11, 32] for the stability criterion in the nonisentropic case.

(2) The stability of shock waves heavily depends on the pressure law, but the general idea is that shock waves of moderate strength are uniformly stable, while large shock waves may be only weakly stable.

Euler equations since it gathers the two main difficulties, namely a characteristic free boundary, and the weak Lopatinskii condition under nonlinear constraints. The main steps of the analysis are outlined so our method can be applied to various situations.

2. The nonlinear equations

We consider the isentropic Euler equations in the whole plane $\mathbb{R}^2$. Denoting by $\mathbf{u} \in \mathbb{R}^2$ the velocity of the fluid, and by ρ its density, the equations read:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \mathbf{u}) = 0, \\ \partial_t (\rho\, \mathbf{u}) + \nabla_x \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla_x\, p = 0, \end{cases} \tag{1}$$

where $p = p(\rho)$ is the pressure law. In all this paper p is a $\mathcal{C}^\infty$ function of ρ, defined on $]0, +\infty[$, and such that $p'(\rho) > 0$ for all ρ. The speed of sound $c(\rho)$ in the fluid is defined by the relation:

$$\forall\, \rho > 0, \quad c(\rho) := \sqrt{p'(\rho)}.$$

It is a well-known fact that, for such a pressure law, (1) is a strictly hyperbolic system in the region $]0, +\infty[\, \times \mathbb{R}^2$, and (1) is also symmetrizable.

In all what follows, the first and second coordinates of the velocity field are denoted respectively v, and u, that is, $\mathbf{u} = (v, u) \in \mathbb{R}^2$. Then, for all $U = (\rho, \mathbf{u}) \in]0, +\infty[\, \times \mathbb{R}^2$, we define the following matrices:

$$A_1(U) := \begin{pmatrix} v & \rho & 0 \\ \dfrac{p'(\rho)}{\rho} & v & 0 \\ 0 & 0 & v \end{pmatrix}, \quad A_2(U) := \begin{pmatrix} u & 0 & \rho \\ 0 & u & 0 \\ \dfrac{p'(\rho)}{\rho} & 0 & u \end{pmatrix}. \tag{2}$$

In the region where $(\rho, \mathbf{u})$ is smooth, (1) is equivalent to the quasilinear equations:

$$\partial_t U + A_1(U)\, \partial_{x_1} U + A_2(U)\, \partial_{x_2} U = 0.$$

In this paper, we are interested in solutions to (1) that are smooth on either side of a surface $\Gamma := \{x_2 = \varphi(t, x_1), t \in [0, T], x_1 \in \mathbb{R}\}$, and such that, at each time $t \in [0, T]$, the tangential velocity is the only quantity that experiments a jump across the curve $\Gamma(t)$. (Tangential should be understood as tangential with respect to $\Gamma(t)$.) The density, and the normal velocity should be continuous across $\Gamma(t)$. For such solutions, the jump conditions across Γ read:

$$\partial_t \varphi = -v^+\, \partial_{x_1} \varphi + u^+ = -v^-\, \partial_{x_1} \varphi + u^-, \quad \rho^+ = \rho^-.$$

As detailed in [12], for the Euler equations (1), these solutions are exactly the contact discontinuities in the sense of Lax [21]. Recall that the second characteristic field of (1) is linearly degenerate, and thus, gives rise to contact discontinuities. For such discontinuous solutions, there is no mass transfer from one side of $\Gamma(t)$ to the other. (Shock waves are exactly the opposite situation where there is a mass transfer from one side to the other.)

The discontinuity surface Γ is part of the unknowns, and it is convenient to reformulate the problem in the fixed domain $\{t \in [0, T], x_1 \in \mathbb{R}, x_2 \geq 0\}$, by introducing a change of variables. This change of variables is detailed in [12, section 2], see also [1, 24, 29]. After *fixing* the unknown front, we are led to constructing smooth solutions $U^\pm = (\rho^\pm, v^\pm, u^\pm)$, $\Phi^\pm$, to the following system of equations:

$$\partial_t U^\pm + A_1(U^\pm)\partial_{x_1} U^\pm + \frac{1}{\partial_{x_2}\Phi^\pm}\left(A_2(U^\pm) - \partial_t \Phi^\pm - \partial_{x_1}\Phi^\pm\, A_1(U^\pm)\right)\partial_{x_2} U^\pm = 0, \tag{3}$$

in the interior domain $\{t \in [0,T], x_1 \in \mathbb{R}, x_2 > 0\}$, with the boundary conditions:

$$\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi, \tag{4a}$$

$$(v^+ - v^-)_{|x_2=0}\, \partial_{x_1}\varphi - (u^+ - u^-)_{|x_2=0} = 0, \tag{4b}$$

$$\partial_t \varphi + v^+_{|x_2=0}\, \partial_{x_1}\varphi - u^+_{|x_2=0} = 0, \tag{4c}$$

$$(\rho^+ - \rho^-)_{|x_2=0} = 0. \tag{4d}$$

We will also consider the initial conditions:

$$(\rho^\pm, v^\pm, u^\pm)_{|t=0} = (\rho_0^\pm, v_0^\pm, u_0^\pm)(x_1,x_2), \quad \varphi_{|t=0} = \varphi_0(x_1). \tag{5}$$

The functions $\Phi^\pm$ should satisfy the constraints:

$$\forall\,(t,x_1,x_2) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+, \quad \pm\partial_{x_2}\Phi^\pm(t,x) \geq \kappa, \tag{6}$$

for a suitable constant $\kappa > 0$, as well as the eikonal equations:

$$\partial_t \Phi^+ + v^+\,\partial_{x_1}\Phi^+ - u^+ = \partial_t \Phi^- + v^-\,\partial_{x_1}\Phi^- - u^- = 0, \tag{7}$$

in the whole domain $\{t \in [0,T], x_1 \in \mathbb{R}, x_2 > 0\}$. Before going on, let us make a few remarks:

REMARK 1. – *The constraint* (6) *ensures that the mapping:*

$$(t,x_1,x_2) \longmapsto \begin{cases} (t,x_1,\Phi^+(t,x_1,x_2)), & \text{if } x_2 > 0,\\ (t,x_1,\Phi^-(t,x_1,-x_2)), & \text{if } x_2 < 0,\end{cases}$$

is a change of variables that straightens the unknown front.

The eikonal equations (7)*, that are clearly imposed on the boundary* $\{x_2 = 0\}$ *by* (4a)-(4b)-(4c)*, ensure that the matrices* $A_2(U^\pm) - \partial_t\Phi^\pm - \partial_{x_1}\Phi^\pm\, A_1(U^\pm)$ *have a constant rank in the whole domain* $\{x_2 \geq 0\}$*, and not only on the boundary. This constant rank property was crucial in* [12] *to perform a Kreiss' type symmetrizers construction and to derive a priori estimates. We refer for instance to* [16, 26, 35, 36] *for various aspects of this constant rank condition in hyperbolic characteristic boundary value problems.*

With an obvious definition, the equations (3) can be rewritten in the compact form:

$$\mathbb{L}(U^+,\Phi^+) = \mathbb{L}(U^-,\Phi^-) = 0. \tag{8}$$

For later use, it is also convenient to write the nonlinear operator $\mathbb{L}$ under the form $\mathbb{L}(U,\Phi) = L(U,\Phi)U$. In other words, we have set:

$$L(U,\Phi)V := \partial_t V + A_1(U)\partial_{x_1} V + \frac{1}{\partial_{x_2}\Phi}\Big(A_2(U) - \partial_t\Phi - \partial_{x_1}\Phi\, A_1(U)\Big)\partial_{x_2} V. \tag{9}$$

In the same way, the boundary conditions (4) can be rewritten in the compact form:

$$\begin{aligned} &\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi,\\ &\mathbb{B}(U^+_{|x_2=0}, U^-_{|x_2=0}, \varphi) = 0. \end{aligned} \tag{10}$$