ISSN 0012-9593

quatrième série - tome 41 fascicule 6 novembre-décembre 2008

ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

Christian BONATTI & Sylvain CROVISIER & Gioia M. VAGO &
Amie WILKINSON

Local density of diffeomorphisms with large centralizers

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4$^{\mathrm{e}}$ série, t. 41, 2008, p. 925 à 954

LOCAL DENSITY OF DIFFEOMORPHISMS WITH LARGE CENTRALIZERS

BY Christian BONATTI, Sylvain CROVISIER, Gioia M. VAGO AND Amie WILKINSON

ABSTRACT. – Given any compact manifold M, we construct a non-empty open subset $\mathcal{O}$ of the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that the centralizer of every diffeomorphism in $\mathcal{D}$ is uncountable, hence non-trivial.

RÉSUMÉ. – Pour toute variété M compacte, de dimension quelconque, nous construisons une partie $\mathcal{O} \subset \mathrm{Diff}^1(M)$ non vide, ouverte dans l'espace $\mathrm{Diff}^1(M)$ des C^1-difféomorphismes de M, et un sous-ensemble $\mathcal{D} \subset \mathcal{O}$ dense en $\mathcal{O}$, constitué de difféomorphismes dont le centralisateur est non dénombrable, donc non trivial.

Introduction

The *centralizer* of a C^r diffeomorphism $f \in \mathrm{Diff}^r(M)$ is the group of diffeomorphisms commuting with f:

$$C(f) := \{g \in \mathrm{Diff}^r(M) : fg = gf\}.$$

The centralizer $C(f)$ always contains the group $\langle f \rangle$ of all the powers of f. For this reason, we say that f has a *trivial centralizer* if $C(f) = \langle f \rangle$. If f is the time one map of a C^r vector field X, then $C(f)$ contains the flow of X and hence contains a subgroup diffeomorphic to $\mathbb{R}$ (or $S^1 = \mathbb{R}/\mathbb{Z}$ if f is periodic).

The elements of $C(f)$ are transformations of M which preserve the dynamics of f: in that sense they are the symmetries of f. How large is, in general, this symmetry group?

– On one hand, the structure on M given by a diffeomorphism is very flexible, so that one might expect that any symmetry could be broken by a small perturbation of the diffeomorphism.

0012-9593/06/

– On the other hand, the symmetries are sought in the very large group $\mathrm{Diff}^r(M)$, which makes the problem harder. For example, one can easily show that the group $C^0(f)$ of homeomorphisms commuting with a Morse-Smale diffeomorphism f is always uncountable.

Nevertheless it is natural to guess that general diffeomorphisms have no non-trivial smooth symmetries. Making this intuition explicit, Smale asked the following:

Question I ([15, 16]). – *Let $\mathcal{T}^r(M) \subset \mathrm{Diff}^{\,r}(M), r \geq 1$, denote the set of C^r diffeomorphisms of a compact manifold M with trivial centralizer.*

1. *Is $\mathcal{T}^r(M)$ dense in* $\mathrm{Diff}^{\,r}(M)$*?*
2. *Is $\mathcal{T}^r(M)$ residual in* $\mathrm{Diff}^{\,r}(M)$*? That is, does it contain the intersection of countably many dense open subsets?*
3. *Is $\mathcal{T}^r(M)$ a dense open subset of* $\mathrm{Diff}^{\,r}(M)$*?*

We think it is natural to reformulate the third part of Smale's question as:

Question II. – *Does $\mathcal{T}^r(M)$ contain a dense and open subset of* $\mathrm{Diff}^{\,r}(M)$*?*

This question has many parameters, the most obvious being the regularity r of the diffeomorphisms and the dimension $\dim(M)$ of the manifold. The question has been answered in varying degrees of generality for specific parameters. For instance, Kopell [12] proved that $\mathrm{Diff}^r(S^1)$, $r \geq 2$, contains a dense and open subset of diffeomorphisms with trivial centralizers. Many authors subsequently gave partial answers in higher dimension (see [5] for an attempt to list these partial results).

The present paper and [7] together give a complete answer to Smale's problem for $r = 1$. More precisely:

– [7] proves that C^1-generic diffeomorphisms have a trivial centralizer [(1)], giving a positive answer to the first two parts of Smale's question. [6] shows that C^1-generic conservative (volume preserving or symplectic) diffeomorphisms have a trivial centralizer.

– In this paper, we answer in the negative (for $r = 1$) the third part of Smale's question (and to Question II) on any compact manifold.

Main theorem. – *Given any compact manifold M, there are a non-empty open subset $\mathcal{O} \subset \mathrm{Diff}^1(M)$ and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ is C^∞ and its C^∞-centralizer $C^\infty(f)$ is uncountable (hence not trivial).*

We will see below (see Theorem 5) that this statement also holds for symplectic diffeomorphisms on a symplectic manifold.

The uniform presentation of this result (*Given any compact manifold,…*) hides very different situations, arguments and results according to the dimension: namely, whether $\dim(M) < 3$ or $\dim(M) \geq 3$. We discuss this breakdown of the results below.

Our paper also deals with the question of how large is the class of diffeomorphisms that can be embedded in a flow. This is a natural question, since the studies of the dynamical

(1) [5] is an announcement which gives the structure of the detailed proof written in [7].

systems defined either by diffeomorphisms or by vector fields are in fact closely related. In the paper [14] titled *Vector fields generate few diffeomorphisms*, Palis proved that C^1-generic diffeomorphisms cannot be embedded in a flow. Our results somehow counterbalance Palis' statement: diffeomorphisms that are the time one map of a flow are C^1-locally dense in dimension 1 and 2.

THEOREM 1. – *There is a dense subset $\mathcal{D} \subset \mathrm{Diff}^1(S^1)$ such that every $f \in \mathcal{D}$ commutes with the flow of a C^∞ Morse-Smale vector field X. More precisely, f is Morse-Smale and f^q is the time one map of the flow of X, where $q = 2$, if f is orientation reversing, and q is the period of the periodic orbits of f otherwise. Furthermore, $C(f)$ is isomorphic to $\mathbb{R} \times (\mathbb{Z}/q\mathbb{Z})$.*

(Section 1.6 presents open questions on centralizers of diffeomorphisms in $\mathrm{Diff}^1(S^1)$.)

Among compact surfaces, the case of the sphere is special because of the existence of north pole-south pole diffeomorphisms. The symmetries of these dynamics allow us to get a centralizer isomorphic to $S^1 \times \mathbb{R}$ and this is one of the reasons why we present this case separately.

Another specific feature of the north-south diffeomorphisms of the sphere is that for these maps it is possible to generalize the so-called *Mather invariant*, introduced by Mather in the one-dimensional case. Such an invariant plays a fundamental role in our constructions: the Mather invariant of a diffeomorphism f is trivial if and only if f can be perturbed to become the time one map of a vector field.

THEOREM 2. – *Let $\mathcal{O} \subset \mathrm{Diff}^1(S^2)$ denote the (open) subset of Morse-Smale diffeomorphisms f such that the nonwandering set $\Omega(f)$ consists of two fixed points, one source N_f and one sink S_f, such that the derivatives $D_{N_f} f$ and $D_{S_f} f$ have each a complex (non-real) eigenvalue.*

Then there is a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ is the time one map of a Morse-Smale C^∞-vector field. Furthermore $C(f)$ is isomorphic to $\mathbb{R} \times S^1$.

Theorem 2 is a bridge between the one-dimensional case and the general two-dimensional case. One the one hand, north pole-south pole dynamics on the sphere and Morse-Smale dynamics on the circle share the Mather invariant; on the other hand, other features of these dynamics on the sphere occur in simple dynamics on a general compact surface. The general case is solved by a combination of the arguments used for the sphere in a neighborhood of the sinks and the sources, together with an analysis of the local situation in a neighborhood of the saddles.

THEOREM 3. – *Let S be a connected closed surface. Let $\mathcal{O} \subset \mathrm{Diff}^1(S)$ be the set of Morse-Smale diffeomorphisms f such that:*

- *any periodic point is a (hyperbolic) fixed point,*
- *f has at least one hyperbolic saddle point,*
- *for any hyperbolic saddle x, every eigenvalue of $Df(x)$ is positive,*
- *for any sink or source x, the derivative $Df(x)$ has a complex (non-real) eigenvalue,*
- *there are no heteroclinic orbits: if $x \neq y$ are saddle points then $W^s(x) \cap W^u(y) = \varnothing$.*

Then $\mathcal{O}$ is a non-empty open subset of $\mathrm{Diff}^1(S)$ and there is a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ is the time one map of a Morse-Smale C^∞-vector field. Furthermore, $C(f)$ is the flow of this vector field, hence isomorphic to $\mathbb{R}$.

An important hypothesis in Theorems 2 and 3 (which holds trivially in Theorem 1) is that the derivative at each sink and source is conjugate to the composition of a homothety with a (non-trivial) rotation. This condition is open in dimension 2, but is nowhere dense in higher dimension. This explains why we are not able to obtain the local density of the embeddability in a flow in higher dimension, and naturally leads to the following question:

QUESTION III. – *Let M be a compact manifold of dimension $d \geq 3$. Denote by $\mathcal{O}$ the C^1-interior of the C^1-closure of the set of diffeomorphisms which are the time one map of a flow. Is $\mathcal{O}$ empty?*

In low dimension we find large centralizers among the simplest dynamical systems (the Morse-Smale systems). By contrast, in higher dimension we will use C^1-open subsets of *wild diffeomorphisms* to obtain periodic islands where the return map is the identity map. The resulting large centralizers for these wild diffeomorphisms are completely different. In low dimension, we embed the diffeomorphisms in a flow, and the centralizer is precisely the flow; hence all the diffeomorphisms in the centralizer have the same dynamics. In higher dimension, the diffeomorphisms we exhibit in the centralizer will be equal to the identity map everywhere but in the islands, in restriction to which they can be anything. This explains our result:

THEOREM 4. – *Given any compact manifold M of dimension $d \geq 3$, there is a non-empty open subset $\mathcal{O} \subset \mathrm{Diff}^1(M)$ and a dense subset $\mathcal{D} \subset \mathcal{O}$ such that every $f \in \mathcal{D}$ has non-trivial centralizer.*

More precisely, for $f \in \mathcal{D}$ the centralizer $C(f)$ contains a subgroup isomorphic to the group $\mathrm{Diff}^1(\mathbb{R}^d, \mathbb{R}^d \setminus \mathbb{D}^d)$ of diffeomorphisms of $\mathbb{R}^d$ which are equal to the identity map outside the unit disc $\mathbb{D}^d$.

The large centralizer we build for a diffeomorphism in Theorem 4 consists of diffeomorphisms which have a very small support, and which are therefore C^0-close to the identity. It would be interesting to know if this is always the case. Let us formalize this question:

QUESTION IV. – *Let M be a compact manifold with $\dim(M) \geq 3$ and $\varepsilon > 0$. Let $\mathcal{O}_\varepsilon \subset \mathrm{Diff}^1(M)$ be the set of diffeomorphisms f such that, for every $g \in C(f)$ there exists $n \in \mathbb{Z}$ such that $g \circ f^n$ is ε-close to the identity map for the C^0-distance. Does $\mathcal{O}_\varepsilon$ contain a dense open subset of $\mathrm{Diff}^1(M)$ for every ε?*

If for non-conservative diffeomorphisms the existence of periodic islands depends on wild dynamics, the same islands appear in a more natural way for symplectic diffeomorphisms in a neighborhood of totally elliptic points. In order to state precisely this last result we need some notations. Let M be a compact manifold with even dimension $\dim(M) = 2d$. If M carries a symplectic form ω, then we denote by $\mathrm{Symp}^1_\omega(M)$ the space of C^1-diffeomorphisms of M that preserve ω (these diffeomorphisms are called *symplectomorphisms*).

Recall that a periodic point x of period n of a symplectomorphism f is called *totally elliptic* if all the eigenvalues of $Df^n(x)$ have modulus equal to 1. If $e^{i\alpha}$ is an eigenvalue of x then $e^{-i\alpha}$ is also an eigenvalue. Assume that $0 < \alpha_1 < \cdots < \alpha_d < \pi$ are the absolute values of the argument of the eigenvalues of x. Then x is C^1-*robustly totally elliptic*: every