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Nader MASMOUDI & Frédéric ROUSSET

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STABILITY OF OSCILLATING BOUNDARY LAYERS IN ROTATING FLUIDS

BY NADER MASMOUDI AND FRÉDÉRIC ROUSSET

ABSTRACT. – We prove the linear and non-linear stability of oscillating Ekman boundary layers for rotating fluids in the so-called ill-prepared case under a spectral hypothesis. Here, we deal with the case where the viscosity and the Rossby number are both equal to ε. This study generalizes the study of [23] where a smallness condition was imposed and the study of [26] where the well-prepared case was treated.

RÉSUMÉ. – On prouve la stabilité linéaire et non-linéaire de couches limites oscillantes de type Ekman pour les fluides tournant dans le cas de données mal préparées sous une hypothèse spectrale. On s'intéresse au cas où la viscosité et le nombre de Rossby sont du même ordre ε. Cette étude généralise celle de [23] où une condition de petitesse était imposée et celle de [26] où les données bien préparées étaient traitées.

1. Introduction

We consider the following system describing the evolution of a rotating fluid in a rectangular domain

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \dfrac{e \times u^\varepsilon}{\varepsilon} + \dfrac{\nabla p}{\varepsilon} - \varepsilon \Delta u^\varepsilon = 0, \\ \nabla \cdot u^\varepsilon = 0 \end{cases} \tag{1}$$

for $x = (y, z) \in \Omega = \mathbb{T}^2_a \times (0, 1)$ with the Dirichlet boundary condition

$$u^\varepsilon_{/\partial\Omega} = 0 \tag{2}$$

and the initial condition

$$u^\varepsilon_{/t=0} = u^{\varepsilon,0}. \tag{3}$$

Here $\mathbb{T}^2_a$ is the periodic torus with periods a_1 and a_2, namely, $\mathbb{T}^2_a = \mathbb{R}^2/(a_1\mathbb{Z} \times a_2\mathbb{Z})$ and $a_1,\ a_2 > 0$. Moreover, $e = e_3$ is the vertical unit vector and $\frac{e \times u^\varepsilon}{\varepsilon}$ is the Coriolis force.

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This system describes the motion of a rotating fluid as the Ekman and Rossby numbers go to zero (see Pedlovsky [25], and Greenspan [14]). It can model the dynamics of the ocean or the atmosphere far from the equator or a rotating fluid in a container. Note that, here, we take the horizontal viscosity and the vertical viscosity to be equal. We point out that in many previous works the horizontal viscosity was supposed constant whereas the vertical viscosity ν goes to 0 (see for instance [17]) or in some other cases, the vertical viscosity was supposed much smaller than the horizontal viscosity. This anisotropy has the advantage of making the boundary layers more stable.

In this paper, we look at the case where the vertical and the horizontal viscosities are equal. We study the convergence of solutions to (1) towards a solution of the limit system (9) defined below once the time oscillations are filtered out.

We recall that this system and related ones were studied by several authors. In the "well-prepared" case in domains with boundary, like Ω, we refer to Colin, Fabrie [5], Grenier, Masmoudi [17], Masmoudi [22]. For general initial data, and for the periodic case, we refer to Grenier [15], Embid and Majda [9], Babin, Mahalov and Nicolaenko [2, 1], Gallagher [12] or in particular cases where there is no boundary layer, or where the boundary layer can be eliminated by symmetry (Beale and Bourgeois [3]). These results rely on the introduction of a group to filter the oscillations in time, a method which was previously used by Schochet [30] to investigate related problems in the torus concerning the compressible-incompressible limit.

In [23], the "group method" was extended to the case of domains with boundary, by solving a superposition of an infinite number of boundary layers. These layers create an extra term in the limit equation. In [23], the stability of these boundary layers was proved in the case where the horizontal viscosity goes to zero slower than the Rossby number (or in the small data case). In this paper, we would like to give a spectral assumption (which we think is optimal) and which yields the stability of such boundary layers.

In the well prepared case, a similar spectral assumption was used to prove the stability of the boundary layer [26]. This spectral assumption is optimal since the instability of the boundary layer was proved in [7] if the spectral assumption does not hold.

In the following sections, we recall the main properties of the approximate solution of (1) constructed in [23] (see also [4]), in particular, we recall the properties of the limit system, of the boundary layers and the assumptions on the torus which are needed. Next, we shall give our main assumption on the spectral stability of the boundary layers and state our main result.

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1.1. Properties of the approximate solution

To state our main result, we first recall the main properties of the approximate solution u^{app} of (1) constructed in [23]. In particular, u^{app} describes the formal limit of (1) and the boundary layers. The details of the construction will be recalled later. The approximate solution is under the form

$$u^{\mathrm{app}} = u^{\mathrm{int}}\left(\frac{t}{\varepsilon}, t, x\right) + u^b\left(\frac{t}{\varepsilon}, \frac{z}{\varepsilon}, \frac{1-z}{\varepsilon}, t, y\right) + u^r, \quad x = (y, z) \in \mathbb{T}_a^2 \times (0,1) \tag{4}$$

where the remainder term u^r satisfies $u^r = \mathcal{O}(\varepsilon)$ (a precise statement will be given later). The interior term u^{int} can be expressed as

$$u^{\mathrm{int}}(\tau, t, x) = \mathcal{L}(\tau) w^{\mathrm{int}}(t, x)$$

where $\mathcal{L}(\tau) = e^{\tau L}$, $Lu = -\mathbb{P}(e \times u)$ and $\mathbb{P}$ is the Leray projector on divergence-free vector fields with zero normal component in Ω. We denote $\mathbb{Z}_a^3 = \frac{2\pi}{a_1}\mathbb{Z} \times \frac{2\pi}{a_2}\mathbb{Z} \times \frac{2\pi}{2}\mathbb{Z}$, $\mathbb{Z}_a^2 = \frac{2\pi}{a_1}\mathbb{Z} \times \frac{2\pi}{a_2}\mathbb{Z}$ and we denote elements of $\mathbb{Z}_a^3$ by $\bar{k} = (k, k_3) \in \mathbb{Z}_a^3$ with $k \in \mathbb{Z}_a^2 = \frac{2\pi}{a_1}\mathbb{Z} \times \frac{2\pi}{a_2}\mathbb{Z}$. We have an expansion

$$w^{\mathrm{int}}(t, x) = \sum_{\bar{k} \in \mathbb{Z}_a^3} b(t, \bar{k}) e^{ik\cdot y} M^{\bar{k}}(z), \tag{5}$$

so that

$$u^{\mathrm{int}}(\tau, t, x) = \mathcal{L}(\tau)(w^{\mathrm{int}}(t, x)) = \sum_{\bar{k} \in \mathbb{Z}_a^3} b(t, \bar{k}) e^{ik\cdot y} M^{\bar{k}}(z) e^{i\lambda(\bar{k})\tau} \tag{6}$$

and w^{int} solves the limit system (9). Note that $N^{\bar{k}} = e^{ik\cdot y} M^{\bar{k}}$ is an eigenvector of L. We assume that the initial data is chosen such that $b(0, (0, k_3)) = 0$ for every k_3 i.e. we exclude initial values with modes which depend only on z. We shall also assume that the torus is non resonant in the sense of [2] to insure that the condition $b(t, (0, k_3)) = 0$ for every k_3 remains true for positive times (see below for a precise definition).

We can express the dominant boundary layer term u^b as

$$u^b(\tau, Z, Z', t, y) = u^{b,0}(\tau, Z, t, y) + u^{b,1}(\tau, Z', t, y)$$

where

$$u^{b,\sigma}(\tau, Z, t, y) = -\frac{1}{2}\sum_{\bar{k}} b(t, \bar{k}) e^{ik\cdot y + i\lambda(\bar{k})\tau}(-1)^{\sigma k_3}\left(h^{\bar{k},+} e^{-\frac{1+i}{\sqrt{2}}\eta^{\bar{k},+} Z} + h^{\bar{k},-} e^{-\frac{1-i}{\sqrt{2}}\eta^{\bar{k},-} Z}\right)$$

for $\sigma = 0,\, 1$, with

$$\eta^{\bar{k},\pm} = \sqrt{1 \pm \lambda(\bar{k})}, \quad h^{\bar{k},\pm} = M^{\bar{k}}(0) \mp ie \times M^{\bar{k}}(0).$$

Note that since terms under the form $(0, k_3)$ are excluded in the above sum, we have $\eta^{\bar{k},\pm} > 0$ and hence, we have a superposition of terms which are small far from the boundary. Nevertheless, the rate of decay, $\eta^{\bar{k},\pm}$, goes to zero when $\frac{k_3}{|\bar{k}|}$ tends to ± 1.

In view of the above definition of the boundary layers, we introduce the operators

$$\mathcal{B}^\sigma(\tau, Z) q = -\frac{1}{2}\sum_{\bar{k}} q_{\bar{k}} e^{i\lambda(\bar{k})\tau}(-1)^{\sigma k_3}\left(h^{\bar{k},+} e^{-\frac{1+i}{\sqrt{2}}\eta^{\bar{k},+} Z} + h^{\bar{k},-} e^{-\frac{1-i}{\sqrt{2}}\eta^{\bar{k},-} Z}\right)$$

for any sequence $q = (q_{\bar{k}})_{\bar{k} \in \mathbb{Z}_a^3}$ so that if q is taken under the form $q_{\bar{k}} = b(t, \bar{k}) e^{ik\cdot y}$, we have $\mathcal{B}^\sigma(\tau, y, Z) q = u^{b,\sigma}(\tau, Z, t, y)$.

In a similar way, we also define

$$\mathcal{L}^\sigma(\tau)q = \sum_{\bar{k}} q_{\bar{k}} M^{\bar{k}}(\sigma) e^{i\lambda(\bar{k})\tau}, \quad \sigma = 0,\, 1.$$

Again, note that if q is taken such that $q_{\bar{k}} = b(t,\bar{k})e^{ik\cdot y}$, then we have $\mathcal{L}^\sigma(\tau)q = w(t,y,\sigma)$.

We shall allways assume that the initial data is sufficiently smooth and vanishes at a sufficient order at $z=0$, $z=1$ in order that $b(t,\bar{k})$ decay to zero sufficiently fast. In particular, we assume that

$$\|w^0\|^2_{V^s_{\rm sym}} = \sum_{\bar{k}\in\mathbb{Z}^3_a} |b(0,\bar{k})|^2 |\bar{k}|^{2s} < \infty \quad \text{for some } s \text{ big enough.} \tag{7}$$

This yields that $w(t) \in V^s_{\rm sym}$ for $0 < t < T^*$ where T^* is the life span of a smooth solution of the limit system (9). Hence, by using that $s > \frac{3}{2} + 2$, we have since

$$\Big(\frac{1}{\eta^{\bar{k},\pm}}\Big)^2 \leq \frac{|\bar{k}|}{|\bar{k}| - |k_3|} = \frac{2|\bar{k}|^2}{k_1^2 + k_2^2} \leq 2|\bar{k}|^2$$

that

$$\sum_{\bar{k}} |b(t,\bar{k})| \Big(1 + \Big(\frac{1}{\eta^{\bar{k},+}}\Big)^2 + \Big(\frac{1}{\eta^{\bar{k},-}}\Big)^2\Big) < \infty$$

to finally obtain the important property

$$\sup_y \int_0^{+\infty} \Big|\partial_Z \mathcal{B}^\sigma(\tau,Z)(w(t,y,\sigma))\Big| (1 + |Z| + |Z^2|)\, dZ < +\infty, \quad \sigma = 0,\, 1 \tag{8}$$

which insures that the boundary layers are sufficiently localized in the vicinity of the boundary.

1.2. The limit system

We denote by $w^{\rm int} = \sum_{\bar{k}\in\mathbb{Z}^3_a} b(t,\bar{k}) N^{\bar{k}}$ the solution in $L^\infty(0,T^*;V^s_{\rm sym})$ of the following system

$$\begin{cases} \partial_t w^{\rm int} + \overline{Q}(w^{\rm int}, w^{\rm int}) + \overline{S}(w^{\rm int}) = -\nabla p \quad \text{in } \Omega, \\ \nabla\cdot w^{\rm int} = 0 \quad \text{in } \Omega, \\ w^{\rm int}\cdot n = \pm w_3 = 0 \quad \text{on } \partial\Omega, \\ w^{\rm int}(t=0) = w^0, \end{cases} \tag{9}$$

where T^* is the time of existence of the smooth solution $w^{\rm int}$ of (9) and $\overline{Q}(w^{\rm int}, w^{\rm int})$, $\overline{S}(w^{\rm int})$ are respectively a bilinear and a linear operator of $w^{\rm int}$.

The bilinear operator is given by

$$\overline{Q}(w^{\rm int}, w^{\rm int}) = \sum_{\substack{\bar{l},\bar{m},\bar{k} \\ \bar{k}\in\mathcal{A}(\bar{l},\bar{m}) \\ \lambda(\bar{l})+\lambda(\bar{m})=\lambda(\bar{k})}} b(t,\bar{l}) b(t,\bar{m}) \alpha_{\bar{l}\bar{m}\bar{k}} N^{\bar{k}}(X). \tag{10}$$

The numbers $\alpha_{\bar{l}\bar{m}\bar{k}}$ are constants and the set $\mathcal{A}(\bar{l},\bar{m}) = \{\bar{l}+\bar{m},\ S\bar{l}+\bar{m},\ \bar{l}+S\bar{m},\ S\bar{l}+S\bar{m}\}$ with the notation

$$S(\bar{l}_1,\bar{l}_2,\bar{l}_3) = (\bar{l}_1,\bar{l}_2,-l_3)$$

is the set of possible resonances.