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LIMITS OF LOG CANONICAL THRESHOLDS

BY Tommaso de FERNEX AND Mircea MUSTAȚĂ

Abstract. – Let $\mathcal{T}_n$ denote the set of log canonical thresholds of pairs (X, Y), with X a nonsingular variety of dimension n, and Y a nonempty closed subscheme of X. Using non-standard methods, we show that every limit of a decreasing sequence in $\mathcal{T}_n$ lies in $\mathcal{T}_{n-1}$, proving in this setting a conjecture of Kollár. We also show that $\mathcal{T}_n$ is closed in $\mathbf{R}$; in particular, every limit of log canonical thresholds on smooth varieties of fixed dimension is a rational number. As a consequence of this property, we see that in order to check Shokurov's ACC Conjecture for all $\mathcal{T}_n$, it is enough to show that 1 is not a point of accumulation from below of any $\mathcal{T}_n$. In a different direction, we interpret the ACC Conjecture as a semi-continuity property for log canonical thresholds of formal power series.

Résumé. – Dans cet article, nous analysons les ensembles $\mathcal{T}_n$ de seuils log canoniques de paires (X, Y), où X est une variété lisse de dimension n, et Y est un sous-schéma fermé non-vide de X. En employant des méthodes non-standard, nous montrons que chaque limite d'une suite strictement décroissante de $\mathcal{T}_n$ appartient à l'ensemble $\mathcal{T}_{n-1}$ (ce résultat a été conjecturé par J. Kollár dans ses travaux sur le sujet). Nous montrons également que l'ensemble $\mathcal{T}_n$ est fermé dans $\mathbf{R}$, et en déduisons que les valeurs adhérentes de l'ensemble des seuils log canoniques des pairs (X, Y) sont rationnelles, si la dimension de X est majorée. Une autre conséquence de nos résultats concerne la conjecture ACC de Shokurov pour les $\mathcal{T}_n$. En effet, nous montrons qu'elle est une conséquence de l'énoncé suivant : pour tout n, la valeur 1 ne peut pas être obtenue comme limite d'une suite strictement croissante de nombres contenus dans $\mathcal{T}_n$. Dans une autre perspective, nous interprétons la conjecture ACC comme une propriété de semi-continuité de seuils log canoniqes des séries formelles.

1. Introduction

Let k be an algebraically closed field of characteristic zero. We consider pairs of the form (X, Y), where X is a smooth variety defined over k and $Y \subseteq X$ is a nonempty closed subscheme. For every integer $n \geq 0$, we are interested in the set of all possible log canonical

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thresholds in dimension n

$$\mathcal{T}_n(k) := \{\mathrm{lct}(X, Y) \mid X \text{ smooth over } k \text{ of dimension } n, \varnothing \neq Y \subseteq X\} \subseteq \mathbf{R},$$

where we make the convention that $\mathrm{lct}(X, X) = 0$. It is well-known that $\mathcal{T}_n(k) \subseteq \mathbf{Q}$. Note that $\mathcal{T}_0(k) = \{0\}$ and $\mathcal{T}_{n-1}(k) \subseteq \mathcal{T}_n(k)$ for every $n \geq 1$.

There are two fundamental questions regarding the accumulation points (in $\mathbf{R}$) of these sets.

CONJECTURE 1.1. – *For every n, the set $\mathcal{T}_n(k)$ has no points of accumulation from below.*

CONJECTURE 1.2. – *For every $n \geq 1$, the set of points of accumulation from above of $\mathcal{T}_n(k)$ is equal to $\mathcal{T}_{n-1}(k)$.*

In particular, the two conjectures predict that every $\mathcal{T}_n(k)$ is closed, and that the set of its accumulation points is equal to $\mathcal{T}_{n-1}(k)$. In fact, both conjectures have stronger formulations, in which $\mathcal{T}_n(k)$ is defined under weaker conditions on the singularities of the ambient variety X.

Conjecture 1.1, known as the *ACC Conjecture*, was formulated by Shokurov in [18], where it was proved for $n = 2$ (in the more general context, alluded to in the previous paragraph). Alexeev proved it for $n = 3$ in [1]. The main interest in this conjecture comes from its implications to the Minimal Model Program, more precisely, to the Termination of Flips Conjecture (see [3] for a precise statement). Conjecture 1.2 above was suggested by Kollár in [11]. It was shown in [15] that Conjecture 1.2 follows if one assumes the Minimal Model Program and a conjecture of Alexeev–Borisov–Borisov on the boundedness of $\mathbf{Q}$-Fano varieties. In particular, it is known to hold (in a more general formulation) for $n \leq 3$.

It is not hard to see that the set $\mathcal{T}_n(k)$ is independent of k (see Propositions 3.1 and 3.3 below). From now on we simply write $\mathcal{T}_n$ instead of $\mathcal{T}_n(k)$. Our main goal is to prove Conjecture 1.2, as well as the fact that $\mathcal{T}_n$ is closed. We state our main results in the following order.

THEOREM 1.3. – *For every n, the set $\mathcal{T}_n$ is closed in $\mathbf{R}$.*

Since $\mathcal{T}_n \subseteq \mathbf{Q} \cap [0, n]$, this immediately implies the following useful property.

COROLLARY 1.4. – *Every limit of log canonical thresholds on smooth varieties of bounded dimension is a rational number.*

THEOREM 1.5. – *For every $n \geq 1$, the set of points of accumulation from above of $\mathcal{T}_n$ is equal to $\mathcal{T}_{n-1}$.*

We mention that there are versions of these results when instead of arbitrary subschemes we consider only hypersurfaces. Suppose that $\mathcal{HT}_n \subseteq \mathcal{T}_n$ is defined by considering only pairs (X, Y), where Y is locally defined by one equation. In this case $\mathcal{HT}_n = \mathcal{T}_n \cap [0, 1]$, hence $\mathcal{HT}_n$ is closed, too, and the set of points of accumulation from above of $\mathcal{HT}_n$ is equal to $\mathcal{HT}_{n-1} \smallsetminus \{1\}$.

Since $\mathcal{HT}_n \subseteq \mathcal{T}_n \subseteq n \cdot \mathcal{HT}_n$, it follows that Conjecture 1.1 holds if and only if, for every n, the set $\mathcal{HT}_n$ has no points of accumulation from below. As a consequence of Corollary 1.4, we show that the conjecture can be reduced to a special case.

COROLLARY 1.6. – *Conjecture 1.1 holds for every n if and only if the following special case holds: for every n, there is $\delta_n \in (0,1)$ such that $\mathcal{HT}_n \cap (\delta_n, 1) = \varnothing$.*

In a different direction, we investigate the ACC Conjecture using the Zariski topology on the set of formal power series.

PROPOSITION 1.7. – *Conjecture 1.1 holds for n if and only if, assuming that k is uncountable, for every c there is an integer $N(n,c)$ such that the condition for f to lie in*

$$\mathcal{R}_n(c) := \{f \in k[\![x_1,\ldots,x_n]\!] \mid f(0) = 0, \mathrm{lct}(f) \geq c\}$$

depends only on the truncation of f up to degree $N(n,c)$.

In fact, we will see that the set $\mathcal{R}_n(c)$ has the property in the above proposition if and only if it is open inside the maximal ideal with respect to the Zariski topology on $k[\![x_1,\ldots,x_n]\!]$ (see §5). Furthermore, Corollary 1.6 implies that in order to prove the ACC Conjecture for every n, it is enough to prove the assertion in the proposition only for the sets $\mathcal{R}_n(1)$.

The main ingredient in the proof of the above theorems is given by non-standard methods. This approach is very natural in this context, when one wants to encode a sequence of polynomials (or ideals) in a single object. In our case, we start with a sequence of ideals $\mathfrak{a}_m \subset k[x_1,\ldots,x_n]$ whose log canonical thresholds converge to some $c \in \mathbf{R}$. Ultrafilter constructions give non-standard extensions of our algebraic structures: we get a field ${}^*k$ containing k and a ring ${}^*(k[x_1,\ldots,x_n])$ containing $k[x_1,\ldots,x_n]$. Moreover, there is a truncation map from ${}^*(k[x_1,\ldots,x_n])$ to the formal power series ring ${}^*k[\![x_1,\ldots,x_n]\!]$. Our sequence of ideals determines an ideal $[\mathfrak{a}_m] \subset {}^*(k[x_1,\ldots,x_n])$ whose image in ${}^*k[\![x_1,\ldots,x_n]\!]$ we denote by $\widetilde{\mathfrak{a}}$. Our key result is that $\mathrm{lct}(\widetilde{\mathfrak{a}}) = c$. After possibly replacing ${}^*k$ by a larger field K, we obtain an ideal in a polynomial ring over K whose log canonical threshold is c. Since $\mathcal{T}_n(k)$ is independent of k, we get the conclusion of Theorem 1.3. If the sequence $\{c_m\}_m$ is strictly decreasing, then we conclude that the limit is actually a log canonical threshold in a smaller dimension via a more careful analysis of the singularities of the ideal $\widetilde{\mathfrak{a}}$. We mention that non-standard methods were also employed in [5] to study the sets of F-pure thresholds of hypersurfaces in positive characteristic (though in that case one could only obtain the analogue of Theorem 1.3 above).

As it should be apparent from the above sketch of the proof, we need to work with log canonical thresholds of ideals in formal power series rings. The familiar framework for studying such invariants is that of schemes of finite type over a field. However, since resolutions of singularities are available for arbitrary excellent schemes (see [19]), it is not hard to extend the theory of log canonical thresholds and multiplier ideals to such a general setting. We explain this extension in detail in the next section.

In §3 we discuss some elementary properties of the sets $\mathcal{T}_n$, in particular the independence of the base field. The proofs of the main results are contained in §4. In §5 we make some comments on Conjecture 1.1, proving Corollary 1.6 and Proposition 1.7. The proposition follows from a basic property of cylinders in the ring of formal power series. This interpretation of the ACC Conjecture illustrates once more that formal power series provide the natural setting when considering sequences of log canonical thresholds.

After the first version of this article was made public, János Kollár gave a new proof of the above theorems using an infinite sequence of approximations and field extensions in place of non-standard methods (at the core the two proofs are the same). In fact, making use of results from [4], he obtains a stronger version of Theorem 1.5, showing that all accumulation points of the set of log canonical thresholds in dimension n (not just the limits of decreasing sequences) are log canonical thresholds in dimension $n-1$.

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2. Multiplier ideals on excellent schemes

Our goal in this section is to develop the theory of multiplier ideals and log canonical thresholds for ideals on a regular excellent scheme of characteristic zero. We will apply this theory when the ambient space is either a smooth scheme of finite type over a field or the spectrum of a formal power series ring over a field. All our schemes have characteristic zero, that is, they are schemes over $\mathrm{Spec}(\mathbf{Q})$.

Recall that a Noetherian ring A is *excellent* if the following hold:

1) For every prime ideal $\mathfrak{p}$ in A, the completion morphism $A_{\mathfrak{p}} \longrightarrow \widehat{A_{\mathfrak{p}}}$ has geometrically regular fibers.
2) For every A-algebra of finite type B, the regular locus of $\mathrm{Spec}(B)$ is open.
3) A is universally catenary.

For the basics on excellent rings we refer to [14]. It is known that every algebra of finite type over an excellent ring is excellent, and that all complete Noetherian local rings are excellent. A Noetherian scheme X is *excellent* if it admits an open cover by spectra of excellent rings.

The key ingredient in building the theory of multiplier ideals is the existence of log resolutions of singularities. It was shown in [19] that Hironaka's Theorem giving existence of resolutions for integral schemes of finite type over a field implies the following general statement (in fact, the result in *loc. cit.* holds for quasi-excellent schemes, but we do not need this generality).

THEOREM 2.1 ([19]). – *Let X be an integral, excellent scheme of characteristic zero, and let $Y \hookrightarrow X$ be a proper closed subscheme. There is a proper, birational morphism $f\colon X' \longrightarrow X$ with the following properties:*

i) *X' is a regular scheme.*
ii) *The inverse image $f^{-1}(Y)$ is a divisor with simple normal crossings.*

Moreover, if $U \subseteq X$ is an open subset of X that is regular, and such that $U \cap Y = \varnothing$, then one can take f to be an isomorphism over U. We note that while the statement in loc. cit. *only gives that $f^{-1}(Y)$ has normal crossings, further resolving to a simple normal crossings divisor is standard.*