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HOFER'S METRICS AND BOUNDARY DEPTH

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ABSTRACT. – We show that if (M, ω) is a closed symplectic manifold which admits a nontrivial Hamiltonian vector field all of whose contractible closed orbits are constant, then Hofer's metric on the group of Hamiltonian diffeomorphisms of (M, ω) has infinite diameter, and indeed admits infinite-dimensional quasi-isometrically embedded normed vector spaces. A similar conclusion applies to Hofer's metric on various spaces of Lagrangian submanifolds, including those Hamiltonian-isotopic to the diagonal in $M \times M$ when M satisfies the above dynamical condition. To prove this, we use the properties of a Floer-theoretic quantity called the boundary depth, which measures the nontriviality of the boundary operator on the Floer complex in a way that encodes robust symplectic-topological information.

RÉSUMÉ. – Nous montrons que si (M, ω) est une variété symplectique fermée qui admet un champ vectoriel hamiltonien non-trivial dont toutes les orbites fermées contractiles sont constantes, la métrique de Hofer sur le groupe des difféomorphismes hamiltoniens de (M, ω) a alors un diamètre infini et admet donc des espaces vectoriels normés plongés quasi-isométriquement et de dimension infinie. Une conclusion semblable s'applique à la métrique de Hofer sur différents espaces de sous-variétés lagrangiennes, y compris les sous-variétés hamiltoniennes isotopiques à la diagonale en $M \times M$ où M satisfait à la condition dynamique ci-dessus. Pour prouver cela, nous utilisons les propriétés d'une quantité Floer-théorique appelée profondeur de bord, qui mesure la non-trivialité de l'opérateur limite sur le complexe de Floer de manière à encoder des informations robustes de topologie symplectique.

1. Introduction

Let (M, ω) be a symplectic manifold and let $H\colon [0,1] \times M \to \mathbb{R}$ be a smooth function, which is compactly supported in $[0,1] \times \mathrm{int}(M)$ in case M is noncompact or has boundary. H then induces a time dependent Hamiltonian vector field by the prescription that

$$\omega(\cdot, X_H(t, \cdot)) = d_M(H(t, \cdot)),$$

and thence an isotopy $\phi_H^t\colon M \to M$ by the prescription that $\phi_H^0 = 1_M$ and $\frac{d}{dt}\phi_H^t(m) = X_H(t, \phi_H^t(m))$.

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The Hamiltonian diffeomorphism group $\mathrm{Ham}(M,\omega)$ is by definition the set of diffeomorphisms $\phi\colon M \to M$ which can be written as $\phi = \phi_H^1$ for some H as above (in particular if M is noncompact or has boundary our convention is that all elements of $\mathrm{Ham}(M,\omega)$ are compactly supported in the interior of M). Of course $\mathrm{Ham}(M,\omega)$ forms a group, all elements of which are symplectomorphisms of (M,ω).

For a function $H\colon [0,1]\times M \to \mathbb{R}$ as above define

$$\mathrm{osc} H = \int_0^1 \left(\max_M H(t,\cdot) - \min_M H(t,\cdot)\right) dt.$$

Now for $\phi \in \mathrm{Ham}(M,\omega)$ let

$$\|\phi\| = \inf\left\{\mathrm{osc} H | \phi_H^1 = \phi\right\}.$$

The *Hofer metric on* $\mathrm{Ham}(M,\omega)$ is then defined by, for $\phi,\psi \in \mathrm{Ham}(M,\omega)$,

$$d(\phi,\psi) = \|\phi^{-1}\circ\psi\|.$$

As was shown for $\mathbb{R}^{2n}$ in [20] and for general symplectic manifolds in [27], d is a nondegenerate, biinvariant metric on $\mathrm{Ham}(M,\omega)$.

Notwithstanding a significant amount of fairly deep work on this metric, our understanding of its global properties remains somewhat limited. In particular, it is not yet known whether the metric is always unbounded. It is widely believed that this is most likely the case, and we provide in this paper further evidence for this belief, as follows:

THEOREM 1.1. – *Suppose that a closed symplectic manifold* (M,ω) *admits a nonconstant autonomous Hamiltonian* $H\colon M \to \mathbb{R}$ *such that all contractible closed orbits of* X_H *are constant. Then the diameter of* $\mathrm{Ham}(M,\omega)$ *with respect to Hofer's metric is infinite. In fact, there is a homomorphism*

$$\Phi\colon \mathbb{R}^\infty \to \mathrm{Ham}(M,\omega)$$

such that, for all $v,w \in \mathbb{R}^\infty$,

$$\|v-w\|_{\ell_\infty} \le d(\Phi(v),\Phi(w)) \le \mathrm{osc}(v-w).$$

Theorem 1.1 is proven in Section 5.2.

To clarify notation, $\mathbb{R}^\infty$ denotes the direct sum of infinitely many copies of $\mathbb{R}$, *i.e.*, the vector space of sequences $\{v_i\}_{i=1}^\infty$ where $v_i \in \mathbb{R}$ and all but finitely many v_i are zero, and for $v = \{v_i\}_{i=1}^\infty$ we write $\mathrm{osc}(v) = \max_{i,j}|v_i - v_j|$ and $\|v\|_{\ell_\infty} = \max_i |v_i|$. Thus $\|v\|_{\ell_\infty} \le \mathrm{osc}(v) \le 2\|v\|_{\ell_\infty}$, and if either all v_i are nonnegative or all v_i are nonpositive then $\|v\|_{\ell_\infty} = \mathrm{osc}(v)$. It will be apparent from the construction that $\Phi(v)$ is generated by a Hamiltonian G_v with $\mathrm{osc} G_v = \mathrm{osc} v$. From this it follows that, for those $v \in \mathbb{R}^\infty$ with $\|v\|_{\ell_\infty} = \mathrm{osc}(v)$, every segment of the path $s \mapsto \Phi(sv)$ minimizes the Hofer length among *all* paths connecting its endpoints. For comparison, there are criteria guaranteeing that a path will be Hofer-length minimizing within its homotopy class in [39], [52] (and our paths do satisfy these criteria), but (except in the rare case that $\mathrm{Ham}(M,\omega)$ is known to be simply connected) it seems to be unusual to find such globally length-minimizing paths in the Hamiltonian diffeomorphism group of a closed symplectic manifold.

To put Theorem 1.1 into context we indicate some examples of symplectic manifolds (M,ω) obeying its hypotheses:

(A) Any positive-genus surface Σ with area form ω admits Hamiltonians as in Theorem 1.1. Indeed if $\gamma \subset \Sigma$ is a noncontractible closed curve and $U \cong \{(s,\theta)|s \in (-\epsilon,\epsilon), \theta \in S^1\}$ is a Darboux-Weinstein neighborhood of γ and if $f\colon (-\epsilon,\epsilon) \to \mathbb{R}$ is a compactly supported smooth function then where $H(s,\theta) = f(s)$ for $(s,\theta) \in U$ and $H(z) = 0$ for $z \notin U$, all orbits of X_H either will be constant or will wrap around a noncontractible loop parallel to γ.

Generalizing this somewhat, consider fiber bundles $\pi\colon M \to \Sigma$ which admit a Thurston-type symplectic form $\Omega = \Omega_0 + K\pi^*\omega$ where Ω_0 is closed and fiberwise symplectic and $K \in \mathbb{R}$. The Ω_0-orthogonal complements to the fibers determine a horizontal subbundle T^hM, and in order to ensure that Ω is symplectic one should take K sufficiently large as to guarantee that at every point it holds that $\Omega|_{T^hM}$ is a positive multiple of the pullback of ω. As long as this condition on K holds, one can check that if $H\colon \Sigma \to \mathbb{R}$ is as in the previous paragraph then $\widetilde{H} = H \circ \pi$ will obey the hypothesis of the theorem, as all orbits γ of $X_{\widetilde{H}}$ which are not constant will be contained in $\pi^{-1}(U)$ and will have $\int_\gamma \pi^* d\theta \neq 0$. Of course this property depends only on the behavior of the symplectic form near $\pi^{-1}(\gamma) \subset M$ and so the property will continue to hold for suitable symplectic forms if instead the map $\pi\colon M \to \Sigma$ has singularities away from γ (*e.g.*, if π is a Lefschetz fibration).

(B) Work of Perutz implies that if Σ is a positive-genus surface and $d \geq 2$ then the symmetric product $M = \mathrm{Sym}^d\Sigma$ obeys the hypothesis of Theorem 1.1, when M is equipped with any of the continuous family of Kähler forms from [46, Theorem A]. Indeed, let $\gamma\colon S^1 \to \Sigma$ be a homologically essential simple closed curve, and let Σ_γ denote the result of surgery along γ (*i.e.*, cut Σ along γ and cap off the resulting boundary components by discs). Perutz then obtains a Lagrangian correspondence $\hat{V}_\gamma \subset \mathrm{Sym}^d\Sigma \times \mathrm{Sym}^{d-1}\Sigma_\gamma$ with the property that the first projection embeds $\hat{V}_\gamma$ as a hypersurface $V_\gamma \subset \mathrm{Sym}^d\Sigma$ while the second projection exhibits $\hat{V}_\gamma$ as a S^1-bundle over $\mathrm{Sym}^{d-1}\Sigma_\gamma$. One can then find a tubular neighborhood $U = (-\epsilon,\epsilon) \times V_\gamma \subset \mathrm{Sym}^d\Sigma$ such that, where s denotes the $(-\epsilon,\epsilon)$ coordinate, a Hamiltonian H which is compactly supported in U and such that $H|_U$ depends only on s will have the property that, at all points, X_H either vanishes or is directed along the fibers of the S^1-bundle $V_\gamma \to \mathrm{Sym}^{d-1}\Sigma_\gamma$. Thus any nonconstant closed orbits of X_H are homotopic to iterates of these S^1 fibers. It follows from [46, Lemma 3.16] that the S^1 fibers are homotopic in $\mathrm{Sym}^d\Sigma$ to loops of the form $t \mapsto \{\gamma(t), p_1, \ldots, p_{d-1}\}$ for any fixed choice of $p_1, \ldots, p_{d-1} \notin \mathrm{Im}(\gamma)$. So the fact that γ is homologically essential in Σ implies (by standard facts about the topology of symmetric products, see *e.g.* the proof of [3, Theorem 9.1]) that the fibers have infinite order in $\pi_1(\mathrm{Sym}^d\Sigma)$. Thus indeed such a Hamiltonian $H\colon \mathrm{Sym}^d\Sigma \to \mathbb{R}$ obeys the requirements of Theorem 1.1.

(C) A variety of symplectic manifolds (M,ω) which admit a nonconstant autonomous Hamiltonian $H\colon M \to \mathbb{R}$ such that X_H has no nonconstant closed orbits at all (contractible or otherwise) are exhibited in [60]. Especially in dimension four, these examples are quite topologically diverse: they include for instance the elliptic surfaces $E(n)$ with $n \geq 2$ as well as infinitely many manifolds homeomorphic but not diffeomorphic to them; the symplectic four-manifolds X_G constructed by Gompf [19] having $\pi_1(X_G) = G$ for any finitely presented group G; and simply-connected symplectic four-manifolds whose Euler characteristics and signatures can be arranged to realize many different values. In general, these examples have a hypersurface $V \subset M$ diffeomorphic to the three-torus such that a suitable Hamiltonian H supported near V will have the property that X_H points along an irrational line on the torus and so has no nonconstant closed orbits. The construction in [60] requires ω to represent an irrational de Rham cohomology class in $H^2(M;\mathbb{R})$; it is not clear whether one can obtain such Hamiltonians when $[\omega]$ is rational.

(D) Obviously, if (M,ω) obeys the hypothesis of Theorem 1.1 then so will $(M \times N, \omega \oplus \sigma)$ for any closed symplectic manifold (N,σ) (regardless of whether (N,σ) obeys the hypothesis). Namely, we can just pull back the Hamiltonian $H\colon M \to \mathbb{R}$ to $M \times N$.

(E) If (M,ω) obeys the hypothesis of Theorem 1.1 and if $(\widetilde{M},\widetilde{\omega})$ is obtained by blowing up a sufficiently small ball $B \subset M$, then $(\widetilde{M},\widetilde{\omega})$ will also obey the hypothesis. For if $H\colon M \to \mathbb{R}$ is as in Theorem 1.1 and if the ball B is small enough that $\overline{H(B)}$ is properly contained in $H(M)$, we can choose a nonconstant smooth function $f\colon H(M) \to \mathbb{R}$ such that $f|_{\overline{H(B)}} = 0$. Then since $X_{f\circ H} = f'(H)X_H$, the vector field $X_{f\circ H}$ will still have no nonconstant contractible closed orbits. But $f \circ H$ now lifts to a Hamiltonian on $\widetilde{M}$, whose Hamiltonian vector field again has no nonconstant contractible closed orbits.

(F) A well-established criterion (used *e.g.* in [30]) for (M,ω) to obey the hypothesis of Theorem 1.1 is for there to exist a Lagrangian submanifold $L \subset M$ such that the inclusion-induced map $\pi_1(L) \to \pi_1(M)$ is injective and such that L admits a Riemannian metric of nonpositive sectional curvature (for in this case the metric on L will have no contractible closed geodesics, and one can take a Hamiltonian supported in a Darboux-Weinstein neighborhood of L which generates a reparametrization of the geodesic flow). Of course the case of a noncontractible closed curve in a surface as in (A) above is a baby example of this. In the presence of such a Lagrangian submanifold, a somewhat weaker version of Theorem 1.1 was proven in [49]—namely Py proves that for all N one has an embedding $\phi\colon \mathbb{Z}^N \to \mathrm{Ham}(M,\omega)$ obeying a bound $C_N^{-1}|v - w|_{\ell_\infty} \le d(\phi(v),\phi(w)) \le C_N|v - w|_{\ell_\infty}$. (Actually, our embedding in Theorem 1.1 appears to reduce to Py's in this special case, and so Theorem 1.1 improves Py's constants.)

It should be clear from the examples that we have provided that the hypothesis of Theorem 1.1 is substantially more general than the assumption that M contains a π_1-injective Lagrangian submanifold which admits a metric with nonpositive sectional curvature. Writing $2n = \dim M$, so that $\dim L = n$, in order for L to admit such a metric L would have to be either flat and hence (by old results of Bieberbach) a finite quotient of T^n, or else by [1, Theorem A] $\pi_1(L)$ would contain a nonabelian free group. Thus $\pi_1(M)$ would have to contain either $\mathbb{Z}^n$ or the free group on two generators. But in many of the above examples $\pi_1(M)$ is not large enough to accommodate such subgroups—indeed in some of the examples M is even simply connected.

There are however closed symplectic manifolds to which Theorem 1.1 can be proven not to apply, namely those which have finite π_1-sensitive Hofer-Zehnder capacity. It is shown in [35, Corollary 1.19] (using an argument that essentially dates back to [22]) that any closed symplectic manifold which admits a nonvanishing genus-zero Gromov-Witten invariant counting pseudoholomorphic spheres that pass through two generic points has finite π_1-sensitive Hofer-Zehnder capacity; if the manifold is simply connected one can instead use arbitrary-genus Gromov-Witten invariants counting curves through two generic points. For instance this applies to all closed toric manifolds (to see this one can use Iritani's theorem [24] that a toric manifold has generically semisimple big quantum homology, so that in particular the class of a point is not nilpotent in quantum homology), and also to any simply-connected closed symplectic four-manifold with $b^+ = 1$ (this follows from work of Taubes and Li-Liu; see [60, Appendix A] for the argument).

There is a substantial history of results showing Hofer's metric on $\mathrm{Ham}(M,\omega)$ to have infinite diameter for a variety of symplectic manifolds (M,ω); Theorem 1.1 overlaps somewhat with these prior results but also includes many new cases (and conversely, there are a some examples which are covered by previous results but are not covered by Theorem 1.1, including $\mathbb{C}P^n$). Notable early results in this direction include those in [28, Section II.5.3], [48], [54, Section 5.1], and [9, Remark 1.10]. More recent work of McDuff [37, Lemma 2.7] shows that the Hofer metric has infinite diameter provided that the asymptotic spectral invariants, which a priori are defined on the universal cover $\widetilde{\mathrm{Ham}}(M,\omega)$, descend to $\mathrm{Ham}(M,\omega)$. [37, Theorems 1.1 and 1.3] provide a range of sufficient conditions for the asymptotic spectral invariants to descend, which are general enough to encompass nearly all of the cases in which infinite Hofer diameter has been proven for closed (M,ω) until now.(1) The argument in [37] combines a construction of Ostrover [45] of a path $\{\phi_t\}_{t\in\mathbb{R}}$ in $\mathrm{Ham}(M,\omega)$ for *any* closed (M,ω) for which the asymptotic spectral invariants (and hence the lifted Hofer pseudonorm on $\widetilde{\mathrm{Ham}}(M,\omega)$) diverge to ∞, with a detailed analysis of the properties of the Seidel representation [55] of $\pi_1(\mathrm{Ham}(M,\omega))$ which finds that the asymptotic spectral invariants descend and hence that Ostrover's path has $\|\phi_t\| \to \infty$ under the conditions given in [37, Theorems 1.1 and 1.3]. Roughly speaking, the hypotheses of [37, Theorems 1.1 and 1.3] ask for (M,ω) to either have large minimal Chern number (at least $n+1$, or n under additional hypotheses, if $\dim M = 2n$) or else to admit few nonvanishing genus zero Gromov-Witten invariants (for instance (M,ω) could be weakly exact or, under mild topological hypotheses, negatively monotone). As is shown in [37], once these conditions are violated the asymptotic spectral invariants can very well fail to descend—for instance by [37, Proposition 1.8] they never descend when (M,ω) is a point blowup of a non-symplectically-aspherical manifold; in this case the minimal Chern number of M can be as large as $n-1$.

There are many manifolds obeying Theorem 1.1 which are not covered by the results of [37] or by any other results on infinite Hofer diameter that I am aware of. For instance McDuff's criteria are not robust under taking products or point blowups, whereas we have noted above that (at least for sufficiently small blowups) the criterion in Theorem 1.1 is preserved under these operations. Thus for instance while the non-symplectically-aspherical *minimal* examples from (C) above obey both Theorem 1.1 and McDuff's criteria, when these

(1) The only exceptions to this that I am aware of are products of positive genus surfaces with other manifolds (for which the result follows from the stabilized non-squeezing theorem of [29], as mentioned on [28, II, p. 64]—of course this case is also covered by Theorem 1.1) and the case of a small blowup of $\mathbb{C}P^2$ which is covered in [36].