ISSN 0012-9593

quatrième série - tome 48 *fascicule 1* *janvier-février 2015*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition / *Publication*

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Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 48, 2015, p. 171 à 226

THE q-ANALOGUE OF THE WILD FUNDAMENTAL GROUP AND THE INVERSE PROBLEM OF THE GALOIS THEORY OF q-DIFFERENCE EQUATIONS

BY JEAN-PIERRE RAMIS AND JACQUES SAULOY

ABSTRACT. – In [23, 24], we defined q-analogues of alien derivations for linear analytic q-difference equations with integral slopes and proved a density theorem (in the Galois group) and a freeness theorem. In this paper, we completely describe the wild fundamental group and apply this result to the inverse problem in q-difference Galois theory.

RÉSUMÉ. – Nous avons défini dans [23, 24] des q-analogues des dérivations étrangères pour les équations aux q-différences linéaires analytiques à pentes entières, et prouvé un théorème de densité (dans le groupe de Galois) et un théorème de liberté. Dans cet article, nous décrivons complètement le groupe fondamental sauvage et appliquons ce résultat au problème inverse en théorie de Galois des équations aux q-différences.

1. Introduction

1.1. The problems

The main purpose of this paper is to give a *new* and probably definitive version of the local meromorphic classification of q-difference modules in the *integral* slopes case(1). Using this result we shall get *a complete solution* of the inverse problem for the q-difference Galois theory in the local case, *for all* $q \in \mathbf{C}^*$, $|q| \neq 1$, and a solution of the inverse problem for *connected reductive algebraic groups* in the global case, also *for all* $q \in \mathbf{C}^*$, $|q| \neq 1$ (for the case of the *exceptional simple groups*, in particular, this result is new(2)).

(1) This is explained in Section 2.2. For the definition and properties of slopes, see Section 2 and [33].
(2) For the simple groups $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{SO}(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$ there are *explicit* solutions with generalized q-hypergeometric difference equations due to J. Roques, cf. Section 5.1

1.1.1. *The q-wild fundamental group*. – In [25] we gave *three* versions of the local meromorphic classification of q-difference modules (in the integral slopes case). The first one uses algebraic normal forms and index theorems, it improves some results of Birkhoff and Guenther [3], there is no analog in the differential case. The second method uses a q-analog of Poincaré asymptotics expansions and the non Abelian cohomology $H^1(\mathbf{E}_q, \Lambda)$ of some sheaves Λ on the (loxodromic) elliptic curve $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$, it parallels some results of Malgrange and Sibuya (after Birkhoff, Balser-Jürkat-Lutz) in the differential case. The third method uses q-multisummability, it parallels [17] in the differential case.

The new version of the classification exposed here is based upon a "fundamental group" $\pi^{(0)}_{1,q,w,1}$ that we named the *q-wild fundamental group*(3), a q-analog of the *wild fundamental group* introduced by the first author in the differential case [7], [17]. There is an equivalence of (Tannakian) categories between the category of finite dimensional representations of this q-wild fundamental group and the category of q-difference modules (with integral slopes), moreover the image of a representation is "the" q-difference Galois group of the corresponding module (see Section 3.6 for a precise definition and statement). This classification is in the style of the *Riemann-Hilbert* correspondence for regular singular meromorphic linear differential equations and should have similar (important…) applications.

Of course there is a "trivial" candidate for a q-wild fundamental group satisfying our requirements: the Tannakian Galois group $\mathrm{Gal}(\mathcal{E}_1^{(0)})$ of the Tannakian category $\mathcal{E}_1^{(0)}$ of our q-modules, but this (proalgebraic) group is "too abstract and too big", our purpose was to get a *smaller* fundamental group (as small as possible !) which is Zariski dense in the Tannakian Galois group and to describe it *explicitly*. (As a byproduct, we shall get finally a complete description of the Tannakian Galois group itself.) It is important to notice that the Tannakian Galois group is an *algebraic object*, but that the construction of the smaller group is based upon *transcendental techniques* (complex analysis). This is similar to what happens with the Riemann-Hilbert correspondance.

We will see that it is possible to write:

$$\mathrm{Gal}(\mathcal{E}_1^{(0)}) = \mathfrak{St} \rtimes \mathrm{Gal}(\mathcal{E}_{p,1}^{(0)})$$

where(4), by definition, $\mathrm{Gal}(\mathcal{E}_{p,1}^{(0)}) := \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*) \times \mathbf{C}$ and $\mathfrak{St}$ is a *prounipotent* group (named *the Stokes group*). We can replace $\mathrm{Gal}(\mathcal{E}_1^{(0)})$ by an equivalent datum, the action of $\mathrm{Gal}(\mathcal{E}_{p,1}^{(0)})$ on the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$. We denote this datum as a semi-direct product $\mathfrak{st} \rtimes \mathrm{Gal}(\mathcal{E}_{p,1}^{(0)})$.

We build a *free* Lie algebra L generated by an *infinite* family of symbols $\dot{\Delta}_i^{(\delta,\bar{c})}$ ($\delta \in \mathbf{N}^*$, $\bar{c} \in \mathbf{E}_q$, $i = 1, \ldots, \delta$) and $\dot{\Delta}^{(0)}$, the (pointed) q-alien derivations, endowed with an action of $\mathrm{Gal}(\mathcal{E}_{p,1}^{(0)})_s := \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$, and a natural $\mathrm{Gal}(\mathcal{E}_{p,1}^{(0)})_s$-equivariant map

(3) In $\pi^{(0)}_{1,q,w,1}$, the subscript 1 is for the analogy with π_1, q is clear, w is for *wild*, the last 1 is for *integral* slopes case (i.e., with denominator 1) and the superscript $^{(0)}$ is for *local at* 0.

(4) This a priori strange notation is motivated by the fact that this group is the Galois group of the category of *pure* modules.

$L \to \tilde{\mathfrak{st}} := \mathfrak{st} \oplus \mathbf{C} \log \dot{\Delta}^{(0)}$. Then, *by definition*:

$$\pi^{(0)}_{1,q,w,1} := L \rtimes \mathrm{Gal}(\mathscr{E}^{(0)}_{p,1})_s$$

and we prove that the natural map

$$\mathrm{Rep}_{\mathbf{C}}(\mathrm{Gal}(\mathscr{E}^{(0)}_1)) \to \mathrm{Rep}_{\mathbf{C}}(\pi^{(0)}_{1,q,w,1})$$

is an *isomorphism*. To be more precise, $\mathrm{Rep}_{\mathbf{C}}(\mathrm{Gal}(\mathscr{E}^{(0)}_1))$ denotes the category of rational finite dimensional complex representations of the proalgebraic group $\mathrm{Gal}(\mathscr{E}^{(0)}_1)$ and $\mathrm{Rep}_{\mathbf{C}}(\pi^{(0)}_{1,q,w,1})$ the category of plain finite dimensional complex representations of the wild fundamental group $\pi^{(0)}_{1,q,w,1}$ (this will be made precise in Definition 3.9). The restriction of representations induces a functor $\mathrm{Rep}_{\mathbf{C}}(\mathrm{Gal}(\mathscr{E}^{(0)}_1)) \to \mathrm{Rep}_{\mathbf{C}}(\pi^{(0)}_{1,q,w,1})$ and this functor is an isomorphism, i.e., it is fully faithful and bijective on objects (see Theorem 3.10). Note that in the text all representations of algebraic or proalgebraic groups will be rational and we shall usually not bother to mention this explicitly.

As a byproduct, we prove that, for some convenient pronilpotent completion $L^\dagger$ (introduced in Section 3.6 and studied in the appendix) of the free Lie-algebra the map:

$$\exp(L^\dagger) \rtimes G^{(0)}_{p,1,s} \to \exp(\tilde{\mathfrak{st}}) \rtimes G^{(0)}_{p,1,s} = \mathfrak{Gt} \rtimes G^{(0)}_{p,1} = G^{(0)}_1$$

is an isomorphism of proalgebraic groups. It is an "explicit description" of the Tannakian group $G^{(0)}_1$.

The construction of L and the proof of its main properties is the outcome of a quite long process (in three steps: [23], [24] and the present article) and uses some deep results of [25]. In [23] we built some (pointed) q-alien derivations $\dot{\Delta}^\delta_a$ belonging to $\mathfrak{st}$[5], we interpreted them using q-Borel-Ramis transform and we got the "first level" of our construction (the "linear case" as in the two-slopes case). In [24] we proved the *Zariski density* of the Lie algebra generated by the q-alien derivations and we gave a first (awkward…) tentative of devissage in order to "free" a convenient *subset* of an extended set of alien derivations. Here we finally give "the good" devissage and we prove the *freeness theorem* (Theorem 3.8). The freeness property is *absolutely crucial*, it allows a very easy *computation* of the representations of the q-wild fundamental group and in particular the solution of the inverse problem.

The (q-Gevrey) devissage used in the present article is based upon the (q-Gevrey) devissage of the non-Abelian cohomology sets of some sheaves of unipotent groups on $\mathbf{E}_q$ and its relations with the q-alien derivations (this is explained in more detail in Sections 3.2 and 3.3). We think that this devissage *is interesting by itself* and will give later some relations between some $H^1(\mathbf{E}_q, \Lambda)$ and some (rational) representations of algebraic groups.

The underlying idea of our construction is that the knowledge of a q-difference module is equivalent to the knowledge of its *formal invariants* and of the corresponding q-Stokes phenomena (in the sense of [25]). This is similar to what happens in the differential case, but unfortunately there is a major difference, here the entries of the Stokes matrices are q-constants, that is elliptic functions on $\mathbf{E}_q$, and we would like instead some matrices belonging to $\mathrm{GL}_n(\mathbf{C})$ (the q-difference Galois groups are defined on $\mathbf{C}$). This motivates the

[5] The pointed q alien derivations are q-analog of the *algebraic* pointed alien derivation introduced in [16]. The name comes from the fact that *in the simplest cases* the Martinet-Ramis pointed alien derivations "coincide" with the derivations introduced before by J. Écalle under this name. For a proof, cf. [15].