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ON THE SPECTRAL THEORY OF GROUPS OF AFFINE TRANSFORMATIONS OF COMPACT NILMANIFOLDS

BY Bachir BEKKA AND Yves GUIVARC'H

Abstract. – Let N be a connected and simply connected nilpotent Lie group, Λ a lattice in N, and $\Lambda\backslash N$ the corresponding nilmanifold. We characterize the countable subgroups of the group $\mathrm{Aff}(\Lambda\backslash N)$ of affine transformations of $\Lambda\backslash N$ whose action on $L^2(\Lambda\backslash N)$ has a spectral gap: these are the groups H for which there exists no proper H-invariant subtorus S of the maximal torus factor T of $\Lambda\backslash N$ such that the projection of H on $\mathrm{Aut}(T/S)$ is a virtually abelian group.

The result is first established when $\Lambda\backslash N$ is a torus. The problem for a general nilmanifold is reduced to the torus case, using Kirillov's theory of unitary representations of nilpotent Lie groups and decay properties of the metaplectic representation of the symplectic group. Our methods show that the action of $H \subset \mathrm{Aff}(\Lambda\backslash N)$ on $\Lambda\backslash N$ is ergodic (or the action of $H \subset \mathrm{Aut}(\Lambda\backslash N)$ on $\Lambda\backslash N$ is strongly mixing) if and only if the corresponding action of H on T has the same property.

Résumé. – Soit N un groupe de Lie nilpotent, connexe et simplement connexe; soient Λ un réseau dans N et $\Lambda\backslash N$ la nilvariété correspondante. Nous donnons une caractérisation des sous-groupes dénombrables du groupe $\mathrm{Aff}(\Lambda\backslash N)$ des transformations affines de $\Lambda\backslash N$ dont l'action sur $L^2(\Lambda\backslash N)$ possède un trou spectral : ce sont les groupes H pour lesquels le tore quotient maximal T de $\Lambda\backslash N$ ne possède aucun sous-tore propre et H-invariant S tel que la projection de H sur $\mathrm{Aut}(T/S)$ soit un groupe virtuellement abélien.

Les outils principaux de la preuve sont la théorie de Kirillov des représentations unitaires des groupes de Lie nilpotents et l'étude du comportement asymptotique des coefficients matriciels de la représentation métaplectique du groupe symplectique qui permettent de ramener le cas général à celui des tores dont l'étude est préalablement menée. Nos méthodes montrent que l'action de $H \subset \mathrm{Aff}(\Lambda\backslash N)$ sur $\Lambda\backslash N$ est ergodique (ou celle de $H \subset \mathrm{Aut}(\Lambda\backslash N)$ sur $\Lambda\backslash N$ est fortement mélangeante) si et seulement si l'action induite de H sur T possède la même propriété.

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1. Introduction

Let H be a countable group acting measurably on a probability space (X,ν) by measure preserving transformations. Let $U : h \mapsto U(h)$ denote the corresponding Koopman representation of H on $L^2(X,\nu)$. We say that the action of H on X has a spectral gap if the restriction U^0 of U to the H-invariant subspace

$$L^2_0(X,\nu) = \{\xi \in L^2(X,\nu) \ : \ \int_X \xi(x)d\nu(x) = 0\}$$

does not have almost invariant vectors, that is, there is no sequence of unit vectors ξ_n in $L^2_0(X,\nu)$ such that $\lim_n \|U^0(h)\xi_n - \xi_n\| = 0$ for all $h \in H$. A useful equivalent condition for the existence of a spectral gap is as follows. Let μ be a probability measure on H and $U^0(\mu)$ the convolution operator defined on $L^2_0(X,\nu)$ by

$$U^0(\mu)\xi = \sum_{h\in H} \mu(h)U^0(h)\xi \qquad \text{for all} \quad \xi \in L^2_0(X,\nu).$$

Observe that we have $\|U^0(\mu)\| \leq 1$ and hence $r(U^0(\mu)) \leq 1$ for the spectral radius $r(U^0(\mu))$ of $U^0(\mu)$. Assume that μ is aperiodic (that is, $\mathrm{supp}(\mu)$ is not contained in the coset of a proper subgroup of H). Then the action of H on X has a spectral gap if and only if $r(U^0(\mu)) < 1$ and this is equivalent to $\|U^0(\mu)\| < 1$.

Ergodic theoretic applications of the existence of a spectral gap (or of the stable spectral gap; see below for the definition) to random walks (such as the rate of L^2-convergence in the random ergodic theorem, pointwise ergodic theorem, analogues of the law of large numbers and of the central limit theorem, etc.) are given in [8], [9], [16], [18] and [19]. Another application of the spectral gap property is the uniqueness of ν as H-invariant mean on $L^\infty(X,\nu)$; for this as well as for further applications, see [5], [31], [39], [42].

Recall that a factor (Y,m,H) of the system (X,ν,H) is a probability space (Y,m) equipped with an H-action by measure preserving transformations together with a H-equivariant mesurable mapping $\Phi : X \to Y$ with $\Phi_*(\nu) = m$. Observe that $L^2(Y,m)$ can be identified with an H-invariant closed subspace of $L^2(X,\nu)$.

By a result proved in [28, Theorem 2.4], no action of a countable amenable group by measure preserving transformations on a non-atomic probability space has a spectral gap. As a consequence, if there exists a non-atomic factor (Y,m,H) of the system (X,ν,H) such that H acts as an amenable group on Y, then the action of H on X has no spectral gap. Our main result (Theorem 1) shows in particular that this is the only obstruction for the existence of a spectral gap when H is a countable group of affine transformations of a compact nilmanifold X.

Let N be a connected and simply connected nilpotent Lie group. Let Λ be a lattice in N; the associated nilmanifold $\Lambda\backslash N$ is known to be compact. The group N acts by right translations on $\Lambda\backslash N$: every $n \in N$ defines a transformation $\rho(n)$ on $\Lambda\backslash N$ given by $\Lambda x \mapsto \Lambda xn$. Denote by $\mathrm{Aut}(N)$ the group of continuous automorphisms of N and by $\mathrm{Aut}(\Lambda\backslash N)$ the subgroup of continuous automorphisms φ of N such that $\varphi(\Lambda) = \Lambda$. The group $\mathrm{Aut}(N)$ is a linear algebraic group defined over $\mathbf{Q}$ and $\mathrm{Aut}(\Lambda\backslash N)$ is a discrete subgroup of $\mathrm{Aut}(N)$. An affine transformation of $\Lambda\backslash N$ is a mapping $\Lambda\backslash N \to \Lambda\backslash N$ of the form $\varphi \circ \rho(n)$ for some $\varphi \in \mathrm{Aut}(\Lambda\backslash N)$ and $n \in N$. The group $\mathrm{Aff}(\Lambda\backslash N)$ of affine transformations of $\Lambda\backslash N$ is the semi-direct product $\mathrm{Aut}(\Lambda\backslash N) \ltimes N$.

Every $g \in \mathrm{Aff}(\Lambda\backslash N)$ preserves the translation invariant probability measure $\nu_{\Lambda\backslash N}$ induced by a Haar measure on N. The action of $\mathrm{Aff}(\Lambda\backslash N)$ on $\Lambda\backslash N$ is a natural generalization of the action of $SL_n(\mathbf{Z}) \ltimes \mathbf{T}^n$ on the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. In fact, let $T = \Lambda[N,N]\backslash N$ be the maximal torus factor of $\Lambda\backslash N$. Then, given a subgroup H of $\mathrm{Aff}(\Lambda\backslash N)$, the nilsystem $(\Lambda\backslash N, H)$ can be viewed as the result, starting with T, of a finite sequence of extensions by tori, with induced actions of H on every stage.

Actions of higher rank lattices by affine transformations on nilmanifolds arise in Zimmer's programme as one of the standard actions for such groups (see the survey [14]). The action of a single affine transformation (or a flow of such transformations) on a nilmanifold have been studied by W. Parry from the ergodic, spectral or topological point of view (see [36], [37], [38]; see also [2] for the case of translations).

Let V be a finite dimensional real vector space and Δ a lattice in V. As is well-known, $T = V/\Delta$ is a torus and Δ defines a rational structure on V. Let W be a rational linear subspace of V. Then $S = W/(W \cap \Delta)$ is a subtorus of T and we have a torus factor $\overline{T} = T/S$. Let H be a subgroup of $\mathrm{Aff}(T)$ and assume that W is invariant under $p_{\mathrm{a}}(H)$, where $p_{\mathrm{a}} : \mathrm{Aff}(T) \to \mathrm{Aut}(T)$ is the canonical projection. Then H leaves S invariant and the induced action of H on $\overline{T}$ is a factor of the action of H on T. We will say that $\overline{T}$ is an H-invariant factor torus of T. Here is our main result.

THEOREM 1. – *Let $\Lambda\backslash N$ be a compact nilmanifold with associated maximal torus factor T. Let H be a countable subgroup* $\mathrm{Aff}(\Lambda\backslash N)$. *The following properties are equivalent:*

(i) *The action of H on $\Lambda\backslash N$ has a spectral gap.*
(ii) *The action of H on T has a spectral gap.*
(iii) *There exists no non-trivial H-invariant factor torus $\overline{T}$ of T such that the projection of $p_{\mathrm{a}}(H)$ on $\mathrm{Aut}(\overline{T})$ is a virtually abelian group (that is, it contains an abelian subgroup of finite index).*

To give an example, let $T = \mathbf{R}^d/\mathbf{Z}^d$ be the d-dimensional torus. Observe that $\mathrm{Aut}(T)$ can be identified with $GL_d(\mathbf{Z})$. Let H be a subgroup of $\mathrm{Aff}(T) = GL_d(\mathbf{Z}) \ltimes T$. Assume that $p_{\mathrm{a}}(H)$ is not virtually abelian and that $p_{\mathrm{a}}(H)$ acts $\mathbf{Q}$-irreducibly on $\mathbf{R}^d$ (that is, there is no non-trivial $p_{\mathrm{a}}(H)$-invariant rational subspace of $\mathbf{R}^d$). Then the action of H on T has a spectral gap. For more details, see Corollary 6 and Example 7 below.

The result above is new even in the case where $\Lambda\backslash N$ is a torus; see however [16, Theorem 6.5.ii] for a sufficient condition for the existence of a spectral gap for groups of torus automorphisms. Our result shows, in particular, that the spectral gap property for a countable subgroup H of $\mathrm{Aff}(\Lambda\backslash N)$ is equivalent to the spectral gap property for its automorphism part $p_{\mathrm{a}}(H)$, where $p_{\mathrm{a}} : \mathrm{Aff}(\Lambda\backslash N) \to \mathrm{Aut}(\Lambda\backslash N)$ is the canonical projection.

The proof of Theorem 1 breaks into two parts. We first establish the result in the case where $\Lambda\backslash N$ is a torus (see Theorem 5 below). Our proof is based here on the existence of appropriate invariant means on finite dimensional vector spaces. A crucial tool will be (a version of) Furstenberg's result on stabilizers of probability measures on projective spaces over local fields. In the case of a general nilmanifold $\Lambda\backslash N$ with associated maximal torus factor T, we show that (ii) implies (i) by studying the asymptotic behavior of matrix coefficients of the Koopman representation U of H restricted to the orthogonal complement of $L^2(T)$