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DENSE FORESTS AND DANZER SETS

BY YAAR SOLOMON AND BARAK WEISS

ABSTRACT. – A set $Y \subseteq \mathbb{R}^d$ that intersects every convex set of volume 1 is called a Danzer set. It is not known whether there are Danzer sets in $\mathbb{R}^d$ with growth rate $O(T^d)$. We prove that natural candidates, such as discrete sets that arise from substitutions and from cut-and-project constructions, are not Danzer sets. For cut and project sets our proof relies on the dynamics of homogeneous flows. We consider a weakening of the Danzer problem, the existence of a uniformly discrete dense forest, and we use homogeneous dynamics (in particular Ratner's theorems on unipotent flows) to construct such sets. We also prove an equivalence between the above problem and a well-known combinatorial problem, and deduce the existence of Danzer sets with growth rate $O(T^d \log T)$, improving the previous bound of $O(T^d \log^{d-1} T)$.

RÉSUMÉ. – Un ensemble de Danzer est une partie Y de $\mathbb{R}^d$ qui rencontre tout ensemble convexe de volume 1. On ne sait pas s'il existe des ensembles de Danzer dans $\mathbb{R}^d$ de croissance $O(T^d)$. Nous démontrons que les candidats naturels, tels que les ensembles discrets produits à l'aide de substitutions, de sections et de projections, ne sont pas des ensembles de Danzer. Dans le cas des sections et projections, notre preuve repose sur la dynamique et la structure des réseaux dans les groupes algébriques. Nous considérons aussi une notion plus faible, l'existence d'une forêt dense uniformément discrète, et nous utilisons la dynamique homogène (en particulier les théorèmes de Ratner sur les flots unipotents) pour construire de tels ensembles. Nous démontrons aussi l'équivalence entre le problème de Danzer et un problème combinatoire classique et en déduisons l'existence d'ensembles de Danzer de croissance $O(T^d \log T)$, améliorant ainsi la borne précédente $O(T^d \log^{d-1} T)$.

1. Introduction

This paper stems from a famous unsolved problem formulated by Danzer in the 1960s (see, e.g., [10, 13, 8, 12]). We will call a subset $Y \subseteq \mathbb{R}^d$ a *Danzer set* if it intersects every convex subset of volume 1. We will say that Y *has growth* $g(T)$, where $g(T)$ is some function, if

$$\#(Y \cap B(0,T)) = O(g(T)) \tag{1.1}$$

(as usual $f(x) = O(g(x))$ means $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ and $B(0,T)$ is the Euclidean ball of radius T centered at the origin in $\mathbb{R}^d$). Danzer asked whether for $d \geq 2$ there is a Danzer set with growth T^d. In this paper we present several results related to this question.

The only prior results on Danzer's question of which we are aware are due to Bambah and Woods. Their paper [5] contains two results. The first is a construction of a Danzer set in $\mathbb{R}^d$ with growth rate $T^d \log^{d-1}(T)$, and the second is a proof that any finite union of grids [(1)] is not a Danzer set. Our paper contains parallel results.

We prove the following theorems. For detailed definitions of the terms appearing in the statements, we refer the reader to the section in which the result is proved.

THEOREM 1.1. – *Let H be a primitive substitution system on the polygonal basic tiles $\{T_1, \ldots, T_n\}$ in $\mathbb{R}^d$. Any Delone set, which is obtained from a tiling $\tau \in X_H$ by picking a point in the same location in each of the basic tiles, is not a Danzer set. Also the set of vertices of tiles in such a tiling is not a Danzer set.*

In particular the vertex set of a Penrose tiling is not a Danzer set. The vertex set of a Penrose tiling has another description, namely as a cut-and-project set. We now consider such sets.

THEOREM 1.2. – *Let Λ be a finite union of cut-and-project sets. Then Λ is not a Danzer set.*

As for positive results, one may try to construct sets which either satisfy a weakening of the Danzer condition, or a weaker growth condition. The following results are in this vein. A set $Y \subseteq \mathbb{R}^d$ is called a *dense forest* if there is a function $\varepsilon(T) \xrightarrow{T\to\infty} 0$ such that for any $x \in \mathbb{R}^d$ and any direction $v \in \mathbb{S}^{d-1} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^d : \|v\| = 1\}$, the distance from Y to the line segment of length T going from x in direction v is at most $\varepsilon(T)$. It is not hard to show that a Danzer set is a dense forest.

THEOREM 1.3. – *Let $U \cong \mathbb{R}^d$ and suppose X is a compact metric space on which U acts smoothly and completely uniquely ergodically. Then for any cross-section $\mathcal{S}$ and any $x_0 \in X$, the set of 'visit times'*

$$\mathcal{D} \stackrel{\text{def}}{=} \{u \in U : u.x_0 \in \mathcal{S}\}$$

is a uniformly discrete set which is a dense forest. In particular, uniformly discrete dense forests exist in $\mathbb{R}^d$ for any d.

By *completely uniquely ergodically* we mean that the restriction of the action to any one-parameter subgroup of U is uniquely ergodic. Our construction of completely uniquely ergodic actions relies on Ratner's theorem and results on the structure of lattices in algebraic groups.

In order to construct Danzer sets which grow slightly faster than $O(T^d)$, we first establish an equivalence between this question and a related finitary question, namely the 'Danzer-Rogers question' (see Question 5.3). We say that a function $g : A \to B$ is *doubling*, where A, B are either $\mathbb{N}$ or $\mathbb{R}$, if there exists some $C > 0$ such that for all $x \in \mathbb{R}$ we have $g(2x) \leq Cg(x)$.

(1) A *grid* is a translated lattice.

THEOREM 1.4. – *For a fixed $d \geq 2$, and a doubling function $g(x)$ that satisfies $\frac{g(x)}{x^d}$ is non-decreasing, the following are equivalent:*

(i) *There exists a Danzer set $Y \subseteq \mathbb{R}^d$ of growth $O(g(T))$.*
(ii) *For every $\varepsilon > 0$ there exists $N_\varepsilon \subseteq [0,1]^d$, such that $\#N_\varepsilon = O(g(\varepsilon^{-1/d}))$, and such that N_ε intersects every box of volume ε in $[0,1]^d$.*

COROLLARY 1.5. – *If $D \subseteq \mathbb{R}^d$ is a Danzer set of growth rate $g(T)$, where $g(x)$ is as in Theorem 1.4, then there exists a Danzer set contained in $\mathbb{Q}^d$ of growth rate $g(T)$.*

Using Theorem 1.4 and known results for the Danzer-Rogers question, we obtain:

THEOREM 1.6. – *There exists a Danzer set in $\mathbb{R}^d$ of growth rate $T^d \log T$.*

Note that for all $d \geq 3$, this improves the result of [5] mentioned above and represents the slowest known growth rate for a Danzer set.

1.1. Structure of the paper

We have attempted to keep the different sections of this paper self-contained. The material on substitution tilings and the cut-and-project sets, in particular the proofs of Theorems 1.1 and 1.2, are contained in § 2 and § 3 respectively. More results from homogeneous flows are used in order to prove Theorem 1.3 in § 4. In § 5 we introduce some terminology from computational geometry and prove Theorem 1.4 and Corollary 1.5. More background from computational geometry and the proof of Theorem 1.6 are in § 6. In § 7 we list some open questions related to the Danzer problem.

1.2. Acknowledgements

The proof of Theorem 1.2 given here relies on a suggestion of Andreas Strömbergsson. Our initial strategy required a detailed analysis of lattices in algebraic groups satisfying some conditions, and an appeal to Ratner's theorem on orbit-closures for homogeneous flows. We reduced the problem to a question on algebraic groups which we were unable to solve ourselves and after consulting with several experts, we received a complete answer from Dave Morris, and his argument appeared in an appendix of the original version of this paper. Later Strömbergsson gave us a simple argument which made it possible to avoid the results of Morris and to avoid Ratner's theorem. The proof which appears here is Strömbergsson's and we are grateful to him for agreeing to include it, and to Dave Morris for his earlier proof. We are grateful to Manfred Einsiedler, Jens Marklof, Tom Meyerovitch, Andrei Rapinchuk, Saurabh Ray, Uri Shapira and Shakhar Smorodinsky for useful discussions. We are also grateful to the referee for a careful reading of our paper. Finally, we are grateful to Michael Boshernitzan for telling us about Danzer's question. We acknowledge the support of ERC starter grant DLGAPS 279893.

2. Nets that Arise from Substitution Tilings

In this section we prove Theorem 1.1, i.e., that primitive substitution tilings do not give rise to Danzer sets. We begin by quickly recalling the basics of the theory of substitution tilings. For further reading we refer to [14, 21, 23, 25].