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HYPERBOLICITY, AUTOMORPHIC FORMS AND SIEGEL MODULAR VARIETIES

BY Erwan ROUSSEAU

Abstract. – We study the hyperbolicity of compactifications of quotients of bounded symmetric domains by arithmetic groups. We prove that, up to a finite étale cover, they are Kobayashi hyperbolic modulo the boundary. Applying our techniques to Siegel modular varieties, we improve some former results of Nadel on the non-existence of certain level structures on Abelian varieties over complex function fields.

Résumé. – Nous étudions l'hyperbolicité des compactifications de quotients de domaines symétriques bornés par des groupes arithmétiques. Nous prouvons, qu'à un revêtement étale fini près, ils sont hyperboliques au sens de Kobayashi modulo le bord. En appliquant ces techniques aux variétés de Siegel modulaires, nous améliorons des résultats antérieurs de Nadel sur la non-existence de certaines structures de niveau pour les variétés abéliennes définies sur des corps de fonctions.

1. Introduction

Any complex space Y can be equipped with an intrinsic pseudo-distance d_Y, the Kobayashi pseudo-distance [6]. It is the largest pseudo-distance such that every holomorphic map $f : \Delta \to Y$ from the unit disc, equipped with the Poincaré metric, is distance decreasing.

Recall [6] that a complex space Y is said to be (Kobayashi) hyperbolic modulo $W \subset Y$ if for every pair of distinct points p, q of Y we have $d_Y(p,q) > 0$ unless both are contained in W. In the case where Y is compact and $W = \varnothing$, Brody's lemma gives a criterion for hyperbolicity: Y is hyperbolic if and only if there are no curves $f : \mathbb{C} \to Y$.

In the case where $W \neq \varnothing$, we do not know an analogue of Brody's criterion for hyperbolicity modulo W. It is not known whether Y is necessarily hyperbolic modulo W if it is

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Brody hyperbolic modulo W i.e., every entire curve $f : \mathbb{C} \to Y$ has image in W. The reason is that the classical Brody's lemma does not provide any information on the image of the entire curve.

In this article, we investigate hyperbolic properties of compactifications of quotients of bounded symmetric domains by arithmetic groups. It is well known that such quotients may be far from being hyperbolic, as rational Hilbert modular surfaces show.

Let $X := \mathbb{B}/\Gamma$ be a quotient of a bounded symmetric domain by an arithmetic subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B})$. By passing to a subgroup of finite index if necessary, we will always suppose that Γ is neat [2]. In particular, X admits a smooth toroidal compactification $\overline{X}$ where $D = \overline{X} \setminus X$ is a normal crossings divisor [1]. Moreover, $\overline{X}$ may be chosen to be projective [1].

We obtain the following result

THEOREM 1.1. – *Let $X := \mathbb{B}/\Gamma$ be a quotient of a bounded symmetric domain by a neat arithmetic subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B})$. Then there exists a finite étale cover $X_1 = \mathbb{B}/\Gamma_1$ determined by a subgroup $\Gamma_1 \subset \Gamma$ such that $\overline{X_1}$, a smooth projective compactification of X_1, is Kobayashi hyperbolic modulo $D_1 := \overline{X_1} \setminus X_1$.*

As a corollary, we obtain a new simple proof of the main result of [8], where it is proved that $\overline{X_1}$ is Brody hyperbolic modulo D_1: the image $f(\mathbb{C})$ of any non-constant holomorphic map $f : \mathbb{C} \to \overline{X_1}$ is contained in D_1.

While Nadel's proof [8] was based on Nevanlinna theory, our proof consists in using the fundamental distance decreasing property of the Kobayashi pseudo-metric mentioned above. To prove Theorem 1.1, we will construct pseudo-distances on $\overline{X_1}$ for which holomorphic maps from the unit disc are distance decreasing.

As an application, we study Siegel modular varieties from this point of view. Their geometry has attracted a lot of attention (see [4] for a survey). In particular, the birational geometry of $\mathscr{A}_g$ and $\mathscr{A}_g(n)$, the moduli spaces of principally polarized Abelian varieties with level structures, has been extensively investigated. The Kodaira dimension of compactifications $\overline{\mathscr{A}_g(n)}$ has been studied by Tai, Freitag, Mumford, Hulek and others, proving the following result.

THEOREM 1.2 (Tai, Freitag, Mumford, Hulek). – *$\overline{\mathscr{A}_g(n)}$ is of general type for the following values of g and $n \geq n_0$:*

g	1	2	3	4	5	6	≥ 7
n_0	7	4	3	2	2	2	1

Therefore it is very natural to study hyperbolicity of these spaces in the light of the Green-Griffiths-Lang conjecture

CONJECTURE 1.3 (Green-Griffiths, Lang). – *Let X be a projective variety of general type. Then there exists a proper algebraic subvariety $Y \subset X$ such that X is hyperbolic modulo Y.*

In this setting, using the theory of automorphic forms, we can apply the same strategy as above and obtain

THEOREM 1.4. – *$\overline{\mathscr{A}_g(n)}$ is hyperbolic modulo $D := \overline{\mathscr{A}_g(n)} \setminus \mathscr{A}_g(n)$ if $n > 6g$.*

As a corollary we have

THEOREM 1.5. – *If A is a principally polarized Abelian variety of dimension $g \geq 1$ which is defined and non-constant over k, a complex function field of genus ≤ 1, then A does not admit a level-n structure for $n > 6g$.*

In particular, this improves the previous bound, $n \geq \max(\frac{1}{2}g(g+1), 28)$, obtained by Nadel in [8].

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2. Distance decreasing pseudo-distances

Let $X := \mathbb{B}/\Gamma$ be a quotient of a bounded symmetric domain by an arithmetic subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B})$. We first give a criterion to ensure the existence of a pseudo-distance $\widetilde{g}$ on $\overline{X}$ non-degenerate at a given $x \in X$ and satisfying the distance decreasing property: every holomorphic map $f : (\Delta, g_P) \to (\overline{X}, \widetilde{g})$ from the unit disc, equipped with the Poincaré metric g_P, is distance decreasing i.e.,

$$f^*\widetilde{g} \leq g_P.$$

Let g denote the Bergman metric on $\mathbb{B}$ satisfying the Kähler-Einstein property $\mathrm{Ric}(g) = -g$. Let $\gamma > 0$ be a rational number such that g has holomorphic sectional curvature $\leq -\gamma$. We want to construct a continuous function $\psi : \overline{X} \to \mathbb{R}^+$ such that $\psi.g$ defines a pseudo-metric on $\overline{X}$.

THEOREM 2.1. – *Let $x \in X$ and suppose there is a section $s \in H^0(\overline{X}, l(K_{\overline{X}} + D))$ such that*

- $s(x) \neq 0$
- *s vanishes on D with multiplicity $m > \frac{l}{\gamma}$.*

Then there exists a pseudo-metric $\widetilde{g}$ on $\overline{X}$ non-degenerate at x satisfying the distance decreasing property. In particular, the Kobayashi pseudo-metric on $\overline{X}$ is non-degenerate at x.

We will prove this theorem in 3 steps.

From [7], we know that the Bergman metric induces a *good* singular metric $h := (\det g)^{-1}$ on $K_{\overline{X}} + D$. Let $x \in X$ and s be as given above. Let $0 < \epsilon$ be such that

$$\gamma - \epsilon > \frac{l}{m}.$$

We define

$$\psi := \|s\|_h^{\frac{2(\gamma-\epsilon)}{l}}.$$

Despite the singularities of h, we have

PROPOSITION 2.2. – *ψ is a continuous function on $\overline{X}$ vanishing on D.*