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LIPSCHITZ STRATIFICATIONS IN O-MINIMAL STRUCTURES

BY Nhan NGUYEN AND Guillaume VALETTE

ABSTRACT. – This paper establishes existence of Lipschitz stratifications in the sense of Mostowski for sets which are definable in a polynomially bounded o-minimal structure. We also improve L. van den Dries and P. Speissegger's preparation theorem for definable functions.

RÉSUMÉ. – Cet article établit l'existence des stratifications lipschitziennes au sens de Mostowski pour les ensembles définissables dans une structure o-minimale polynomialement bornée. On améliore aussi le théorème de préparation de L. van den Dries et P. Speissegger.

Introduction

Stratifications naturally appear in many contexts of modern geometry. They are needed to perform differential geometry on singular sets, to prove stability theorems, or to establish finiteness properties. Recall that a stratification of a set $X \subset \mathbb{R}^n$ is a locally finite partition of X into smooth submanifolds of $\mathbb{R}^n$, called strata. We often generally require some extra conditions on the strata in order to describe the way these sets glue together. The most famous regularity conditions for stratifications are the Whitney's conditions (a) and (b). One can prove that many sets occurring in algebraic or analytic geometry, such as semi-algebraic or subanalytic sets, do admit Whitney stratifications [16, 2, 7]. Whitney's (b) condition turned out to have many properties. It was used by R. Thom and then J. Mather to establish the now famous isotopy lemmas.

The first Thom-Mather isotopy lemma ensures that if X has a Whitney (b) regular stratification and if $f : X \to \mathbb{R}^p$ is a proper continuous map which induces a submersion on every stratum then f is a topologically trivial fibration. This is a generalization of Ehresmann's theorem to singular sets.

The topological equivalence considered in Thom-Mather isotopy lemma is often too weak to investigate the geometry of singular sets. It was also observed that C^1 equivalence is too strong to investigate the stability of singularities since it admits continuous moduli even in

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the algebraic category. People therefore set about investigating an intermediate equivalence relation: the bi-Lipschitz equivalence. Metric stability naturally appeared as an intermediate notion between C^1 and topological equivalence. Bi-Lipschitz equivalence provides a much more accurate information than its topological counterpart. For instance, bi-Lipschitz maps preserve the Hausdorff dimension and negligible sets.

In order to study the singularities from the metric point of view, T. Mostowski introduced *the Lipschitz stratifications* [19]. These stratifications satisfy a bi-Lipschitz version of first Thom-Mather isotopy lemma (Theorem 2.8). T. Mostowski also established that every complex analytic set can be stratified in this way. Existence of Lipschitz stratifications was then extended to the (real) semi-analytic and subanalytic sets by A. Parusiński ([20] [22]). We show in this paper that every set which is definable in a polynomially bounded o-minimal structure admits a Lipschitz stratification (Theorem 2.6). This generalizes Parusiński's theorem to a much wider class of sets enclosing, for instance, all the sets which are definable in the quasi-analytic Denjoy-Carleman classes [25]. O-minimal structures have recently been proved to have many applications to analysis. Their study from the metric point of view is hence definitely of interest and valuable for applications.

Existence of Lipschitz stratifications for globally subanalytic sets was used for instance in [3] in order to establish that the set of parameters at which the fibers of a globally subanalytic family have finite volume is globally subanalytic. The argument used in the latter article indeed also applies to any o-minimal structure that admits Lipschitz stratifications. Theorem 2.6 therefore makes it possible to extend this result to the o-minimal framework (polynomially bounded). For families of surfaces, this result was obtained by T. Kaiser [8] without using Lipschitz stratifications.

Bi-Lipschitz triviality of families that are definable in a polynomially bounded o-minimal structure was proved by the second author without using integration of vector fields [26].

The polynomially bounded o-minimal structures are those which satisfy the so-called Łojasiewicz inequality. These categories of sets can thus be considered as generalizations of the semi-algebraic and subanalytic sets.

If it is well known that sets which are definable in an o-minimal structure (polynomially bounded or not) admit Whitney regular stratifications [14, 15], it was however unclear whether they admit Lipschitz stratifications. If the structure is not polynomially bounded then it is possible to show that there is a definable set for which the bi-Lipschitz version of Thom-Mather isotopy lemma (Theorem 2.8) does not hold (for any stratification of this set, see Example 2.9). Consequently, Lipschitz stratifications do not always exist for definable sets if the o-minimal structure is not required to be polynomially bounded. This is the reason why this work definitely settles the issue of the existence of Lipschitz stratifications for sets which are definable in an o-minimal structure expanding the real field.

The main ingredient of A. Parusiński's proof of existence of Lipschitz stratifications for subanalytic sets [22] is the Preparation Theorem (see also [13]). This theorem offers a nice description of subanalytic functions (up to a partition) in terms of convergent series. This statement unfortunately no longer holds true in the o-minimal framework. It is even actually unclear whether the quasi-analytic o-minimal structures [25] satisfy a preparation theorem as in [13, 22]. In [7], the authors have proved an o-minimal version of the preparation theorem but this statement does not provide estimates on the partial derivatives of the unit (see

Definition 3.2, see also Remark (1) after Theorem 2.1 of [7]). One of the major difficulties of the proof was therefore to achieve an adequate version of the preparation theorem. In this article, we improve L. van den Dries and P. Speissegger's preparation theorem by showing that the unit can be expressed as a composite of a map with bounded derivative together with a map of the same form as in the subanalytic setting (see [13, 22]). This result which is of its own interest is one the key ingredients of the proof of existence of Lipschitz stratifications for definable sets.

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1. Lipschitz stratifications in o-minimal structures

We start by recalling the notion of o-minimal structure. For a more detailed introduction on the subject, we refer the reader to [6, 4].

1.1. O-minimal structures

A *structure* on an ordered field $(\mathbb{R}, +, .)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ such that for each n the following properties hold

(1) $\mathcal{D}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$.
(2) If $A \in \mathcal{D}_n$ then $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{D}_{n+1}$.
(3) $\mathcal{D}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$, where $P \in \mathbb{R}[X_1, \dots, X_n]$.
(4) If $A \in \mathcal{D}_n$ then $\pi(A)$ belongs to $\mathcal{D}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the standard projection onto the first $(n-1)$ coordinates.

Such a family $\mathcal{D}$ is said to be *o-minimal* if in addition:

(5) Any set $A \in \mathcal{D}_1$ is a finite union of intervals and points.

A set belonging to the structure $\mathcal{D}$ is called a *$\mathcal{D}$-set* (or a *definable set*) and a map whose graph is in the structure $\mathcal{D}$ is called a *$\mathcal{D}$-map* (or a *definable map*).

A structure $\mathcal{D}$ is said to be *polynomially bounded* if for each $\mathcal{D}$-function $f : \mathbb{R} \to \mathbb{R}$, there exists a positive number a and an $n \in \mathbb{N}$ such that $|f(x)| < x^n$ for all $x > a$.

Examples of polynomially bounded o-minimal structures are the semi-algebraic sets, the globally subanalytic sets [5, 13] but also the so called x^λ-sets [18, 13] as well as the structures defined by the Denjoy-Carleman classes of functions [25].

Let $p \in \mathbb{N}$. We say that a subset C of $\mathbb{R}^n$ is a *C^p $\mathcal{D}$-cell* if

$n = 1$: C is either a point or an open interval.
$n > 1$: C is of one of the following forms

$$\begin{aligned}
\Gamma_\xi &:= \{(x, y) \in B \times \mathbb{R} : y = \xi(x)\},\\
(\xi_1, \xi_2) &:= \{(x, y) \in B \times \mathbb{R} : \xi_1(x) < y < \xi_2(x)\},\\
(-\infty, \xi) &:= \{(x, y) \in B \times \mathbb{R} : y < \xi(x)\}\\
(\xi, +\infty) &:= \{(x, y) \in B \times \mathbb{R} : \xi(x) < y\},
\end{aligned}$$