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The universal abelian variety over $\mathbb{A}_5$
THE UNIVERSAL ABELIAN VARIETY OVER $\mathcal{A}_5$

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Abstract. — We establish a structure result for the universal abelian variety over $\mathcal{A}_5$. This implies that the boundary divisor of $\overline{\mathcal{A}}_6$ is unirational and leads to a lower bound on the slope of the cone of effective divisors on $\overline{\mathcal{A}}_6$.

Résumé. — On établit un théorème de structure pour la variété abélienne universelle sur $\mathcal{A}_5$. Le résultat entraîne que le diviseur de la frontière de $\overline{\mathcal{A}}_6$ est unirationnel et ceci donne lieu à une borne inférieure pour la pente du cône des diviseurs effectifs en $\overline{\mathcal{A}}_6$.

The general principally polarized abelian variety $[A, \Theta] \in \mathcal{A}_g$ of dimension $g \leq 5$ can be realized as a Prym variety. Abelian varieties of small dimension can be studied in this way via the rich and concrete theory of curves, in particular, one can establish that $\mathcal{A}_g$ is unirational in this range. In the case $g = 5$, the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ is finite of degree 27, see [7]; three different proofs [6, 17], [22] of the unirationality of $\mathcal{R}_6$ are known. The moduli space $\mathcal{A}_g$ is of general type for $g \geq 7$, see [12, 18], [21]. Determining the Kodaira dimension of $\mathcal{A}_6$ is a notorious open problem.

The aim of this paper is to give a simple unirational parametrization of the universal abelian variety over $\mathcal{A}_5$ and hence of the boundary divisor of a compactification of $\mathcal{A}_6$. We denote by $\phi : \mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ the universal abelian variety of dimension $g - 1$ (in the sense of stacks). The moduli space $\overline{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension $g$ and their rank 1 degenerations is a partial compactification of $\mathcal{A}_g$ obtained by blowing up $\mathcal{A}_{g-1}$ in the Satake compactification, cf. [18]. Its boundary $\partial \overline{\mathcal{A}}_g$ is isomorphic to the universal Kummer variety in dimension $g - 1$ and there exists a surjective double covering $j : \mathcal{X}_{g-1} \to \partial \overline{\mathcal{A}}_g$. We establish a simple structure result for the boundary $\partial \overline{\mathcal{A}}_6$:

Theorem 0.1. — The universal abelian variety $\mathcal{X}_5$ is unirational.

This immediately implies that the boundary divisor $\partial \overline{\mathcal{A}}_6$ is unirational as well. What we prove is actually stronger than Theorem 0.1. Over the moduli space $\mathcal{R}_g$ of smooth Prym curves of genus $g$, we consider the universal Prym variety $\varphi : \mathcal{Y}_g \to \mathcal{R}_g$ obtained by pulling back $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ via the Prym map $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Let $\overline{\mathcal{R}}_g$ be the moduli space of stable
Prym curves of genus $g$ together with the universal Prym curve $\pi : \tilde{\mathcal{C}} \to \mathcal{R}_g$ of genus $2g - 1$. If $\tilde{\mathcal{C}}^{g-1} := \tilde{\mathcal{C}} \times_{\mathcal{R}_g} \cdots \times_{\mathcal{R}_g} \tilde{\mathcal{C}}$ is the $(g - 1)$-fold product, one has a universal Abel-Prym rational map $\varphi : \tilde{\mathcal{C}}^{g-1} \to \mathcal{R}_1$ whose restriction on each individual Prym variety is the usual Abel-Prym map. The rational map $\varphi$ is dominant and generically finite (see Section 4 for details). We prove the following result:

**Theorem 0.2.** — The five-fold product $\tilde{\mathcal{C}}^5$ of the universal Prym curve over $\mathcal{R}_6$ is unirational.

The key idea in the proof of Theorem 0.2 is to view smooth Prym curves of genus 6 as discriminants of conic bundles, via their representation as symmetric determinants of quadratic forms in three variables. We fix four general points $u_1, \ldots, u_4 \in \mathbb{P}^2$ and set $w_i := (u_i, u_i) \in \mathbb{P}^2 \times \mathbb{P}^2$. Since the action of the automorphism group $\text{Aut}(\mathbb{P}^2 \times \mathbb{P}^2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ is 4-transitive, any set of four general points in $\mathbb{P}^2 \times \mathbb{P}^2$ can be brought to this form. We then consider the linear system

$$\mathbb{P}^{15} := \left| \mathfrak{S}^2_{\{w_1, \ldots, w_4\}}(2, 2) \right| \subset \left| \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2, 2) \right|$$

of hypersurfaces $Q \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$ which are nodal at $w_1, \ldots, w_4$. For a general threefold $Q \subset \mathbb{P}^{15}$, the first projection $p : Q \to \mathbb{P}^2$ induces a conic bundle structure with a sextic discriminant curve $\Gamma \subset \mathbb{P}^2$ such that $p(\text{Sing}(Q)) = \text{Sing}(\Gamma)$. The discriminant curve $\Gamma$ is nodal precisely at the points $u_1, \ldots, u_4$. Furthermore, $\Gamma$ is equipped with an unramified double cover $p_{\Gamma} : \tilde{\Gamma} \to \Gamma$, parametrizing the lines which are components of the singular fibres of $p : Q \to \mathbb{P}^2$. By normalizing, $p_{\Gamma}$ lifts to an unramified double cover $f : \tilde{C} \to C$ between the normalization $\tilde{C}$ of $\Gamma$ and the normalization $C$ of $\Gamma$ respectively. Note that there exists an exact sequence of generalized Prym varieties

$$0 \longrightarrow (C^*)^4 \longrightarrow P(\tilde{\Gamma}/\Gamma) \longrightarrow P(\tilde{C}/C) \longrightarrow 0.$$

One can show without much effort that the assignment $\mathbb{P}^{15} \not
Q : \{ \tilde{C} \to C \} \subset \mathcal{R}_6$ is dominant. This offers an alternative, much simpler, proof of the unirationality of $\mathcal{R}_6$. However, much more can be obtained with this construction.

Let $G := \mathbb{P}^2 \times (\mathbb{P}^2)^\vee = \left\{ (o, \ell) : o \in \mathbb{P}^2, \ell \in \{ o \} \times (\mathbb{P}^2)^\vee \right\}$ be the Hilbert scheme of lines in the fibres of the first projection $p : \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \to \mathbb{P}^2$. Since containing a given line in a fibre of $p$ imposes three linear conditions on the linear system $\mathbb{P}^{15}$ of threefolds $Q \subset \mathbb{P}^2 \times \mathbb{P}^2$ as above, it follows that imposing the condition $\{ o_i \} \times \ell_i \subset Q$ for five general lines, singles out a unique conic bundle $Q \subset \mathbb{P}^{15}$. This induces an étale double cover $f : \tilde{C} \to C$, as above, over a smooth curve of genus 6. Moreover, $f$ comes equipped with five marked points $\ell_1, \ldots, \ell_5 \in \tilde{C}$. To summarize, we can define a rational map

$$\zeta : G^5 \longrightarrow \tilde{\mathcal{C}}^5, \quad \zeta((o_1, \ell_1), \ldots, (o_5, \ell_5)) := \left( f : \tilde{C} \to C, \ell_1, \ldots, \ell_5 \right),$$

between two 20-dimensional varieties, where $G^5$ denotes the 5-fold product of $G$.

**Theorem 0.3.** — The morphism $\zeta : G^5 \longrightarrow \tilde{\mathcal{C}}^5$ is dominant, so that $\tilde{\mathcal{C}}^5$ is unirational.
More precisely, we show that $G^5$ is birationally isomorphic to the fibre product $P^{15} \times \widetilde{C}^5$. In order to set Theorem 0.3 on the right footing and in view of enumerative calculations, we introduce a $P^2$-bundle $\pi : P(M) \to S$ over the quintic del Pezzo surface $S$ obtained by blowing up $P^2$ at the points $u_1, \ldots, u_4$. The rank 3 vector bundle $M$ on $S$ is obtained by making an elementary transformation along the four exceptional divisors $E_1, \ldots, E_4$ over $u_1, \ldots, u_4$. The nodal threefolds $Q \subset P^2 \times P^2$ considered above can be thought of as sections of a tautological linear system on $P(M)$ and, via the identification

$$\left| \mathcal{I}_{w_1, \ldots, w_4} \right|^2 (2, 2) = \left| \mathcal{O}_{P(M)}(2) \right|,$$

we can view 4-nodal conic bundles in $P^2 \times P^2$ as smooth conic bundles over $S$. In this way the numerical characters of a pencil of such conic bundles can be computed (see Sections 2 and 3 for details).

Theorem 0.3 is then used to give a lower bound for the slope of the effective cone of $\overline{\mathcal{M}}_0$ (though we stop short of determining the Kodaira dimension of $\overline{\mathcal{M}}_0$). Recall that if $E$ is an effective divisor on the perfect cone compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ with no component supported on the boundary $D_g := \overline{\mathcal{M}}_g - \mathcal{M}_g$ and $[E] = a\lambda_1 - b[D_g]$, where $\lambda_1 \in CH^1(\overline{\mathcal{M}}_g)$ is the Hodge class, then the slope of $E$ is defined as $s(E) := \frac{b}{a} \geq 0$. The slope $s(\overline{\mathcal{M}}_g)$ of the effective cone of divisors of $\overline{\mathcal{M}}_g$ is the infimum of the slopes of all effective divisors on $\overline{\mathcal{M}}_g$. This important invariant governs to a large extent the birational geometry of $\overline{\mathcal{M}}_g$; for instance, since $K_{\mathcal{M}_g} = (g + 1)\lambda_1 - [D_g]$, the variety $\overline{\mathcal{M}}_g$ is of general type if $s(\overline{\mathcal{M}}_g) < g + 1$, and uniruled when $s(\overline{\mathcal{M}}_g) > g + 1$. It is shown in the appendix of [14] that the slope of the moduli space $\overline{\mathcal{M}}_g$ is independent of the choice of a toroidal compactification.

It is known that $s(\overline{\mathcal{M}}_4) = 8$ and that the Jacobian locus $\mathcal{M}_4 \subset \overline{\mathcal{M}}_4$ achieves the minimal slope [19]; one of the results of [9] is the calculation $s(\overline{\mathcal{M}}_4) = \frac{54}{10}$. Furthermore, the only irreducible effective divisor on $\overline{\mathcal{M}}_5$ of minimal slope is the closure of the Andreotti-Mayer divisor $\mathcal{N}_g^5$ consisting of 5-dimensional ppav's $[A, \Theta]$ for which the theta divisor $\Theta$ is singular at a point which is not 2-torsion. Concerning $\overline{\mathcal{M}}_6$, we establish the following estimate:

**Theorem 0.4.** -- The following lower bound holds: $s(\overline{\mathcal{M}}_6) \geq \frac{54}{10}$.

Note that this is the first concrete lower bound on the slope of $\overline{\mathcal{M}}_6$. The idea of proof of Theorem 0.4 is very simple. Since $\widetilde{C}^5$ is unirational, we choose a suitable sweeping rational curve $i : \mathbb{P}^1 \to \widetilde{C}^5$, which we then push forward to $\overline{\mathcal{M}}_6$ as follows:

$$\mathbb{P}^1 \stackrel{i}{\longleftarrow} \widetilde{C}^5 \stackrel{ap}{\rightarrow} \overline{\mathcal{Y}}_6 \longrightarrow \overline{\mathcal{X}}_5 \longrightarrow D_6.$$

Here $\overline{\mathcal{Y}}_6$ and $\overline{\mathcal{X}}_5$ are partial compactifications of $\mathcal{Y}_6$ and $\mathcal{X}_5$ respectively which are described in Section 4, whereas $D_6$ is the boundary divisor of $\overline{\mathcal{M}}_6$. The curve $h(\mathbb{P}^1)$ sweeps the boundary divisor of $\overline{\mathcal{M}}_6$ and intersects non-negatively any effective divisor on $\overline{\mathcal{M}}_6$ not containing $D_6$. In particular,

$$s(\overline{\mathcal{M}}_6) \geq \frac{h(\mathbb{P}^1) \cdot [D_6]}{h(\mathbb{P}^1) \cdot \lambda_1}.$$

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