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O-MINIMALITY AND CERTAIN ATYPICAL INTERSECTIONS

BY Philipp HABEGGER AND Jonathan PILA

ABSTRACT. – We show that the strategy of point counting in o-minimal structures can be applied to various problems on unlikely intersections that go beyond the conjectures of Manin-Mumford and André-Oort. We verify the so-called Zilber-Pink Conjecture in a product of modular curves on assuming a lower bound for Galois orbits and a sufficiently strong modular Ax-Schanuel Conjecture. In the context of abelian varieties we obtain the Zilber-Pink Conjecture for curves unconditionally when everything is defined over a number field. For higher dimensional subvarieties of abelian varieties we obtain some weaker results and some conditional results.

RÉSUMÉ. – On démontre que la stratégie de comptage dans des structures o-minimales est suffisante pour traiter plusieurs problèmes qui vont au-delà des conjectures de Manin-Mumford et André-Oort. On vérifie la conjecture de Zilber-Pink pour un produit de courbes modulaires en supposant une minoration assez forte pour la taille de l'orbite de Galois et en supposant une version modulaire du théorème de Ax-Schanuel. Dans le cas des variétés abéliennes, on démontre la conjecture de Zilber-Pink pour les courbes si tous les objets sont définis sur un corps de nombres. Pour les sous-variétés de dimension supérieure, on obtient quelques résultats plus faibles et quelques résultats conditionnels.

1. Introduction

The object of this paper is to show that the "o-minimality and point-counting" strategy can be applied to quite general problems of "unlikely intersection" type as formulated in the Zilber-Pink Conjecture (ZP; see Section 2 for various formulations), provided one assumes certain arithmetic and functional transcendence hypotheses. In these problems there is an ambient variety X of a certain type equipped with a distinguished collection $\mathcal{S}$ of "special" subvarieties. The conjecture governs the intersections of a subvariety $V \subseteq X$ with the members of $\mathcal{S}$. In the problems we consider, X will be either a product of non-compact modular curves (for which it is sufficient to consider the case $X = Y(1)^n$) or an abelian variety, but the same formulations should be applicable more generally. In this paper, a subvariety is always geometrically irreducible and therefore in particular non-empty. A curve is a subvariety of dimension 1.

Our most general results are conditional, but let us state first an unconditional result in the abelian setting.

Say X is an abelian variety defined over a field K and $\overline{K}$ is a fixed algebraic closure of K. For any $r \in \mathbb{R}$ we write

$$X^{[r]} = \bigcup_{\operatorname{codim}_X H \geq r} H(\overline{K})$$

where H runs over algebraic subgroups of X satisfying the dimension condition.

THEOREM 1.1. – *Let X be an abelian variety defined over a number field K and suppose $V \subseteq X$ is a curve, also defined over K. If V is not contained in a proper algebraic subgroup of X, then $V(\overline{K}) \cap X^{[2]}$ is finite.*

This theorem is the abelian version of Maurin's Theorem [29]. We will see a more precise version in Theorem 9.14.

We briefly describe previously known cases of Theorem 1.1 under additional hypotheses on V or X. Viada [49] proved finiteness for V not contained in the translate of a proper abelian subvariety and if X is the power of an elliptic curve with complex multiplication. Rémond and Viada [45] then removed the hypothesis on V. This was later generalized by Ratazzi [40] to when the ambient group variety is isogenous to a power of an abelian variety with complex multiplication. Carrizosa's height lower bound [12, 13] in combination with Rémond's height upper bound [43] led to a proof for all abelian varieties with complex multiplication. Work of Galateau [18] and Viada [50] covers the case of an arbitrary product of elliptic curves.

More generally we show, in the abelian and modular settings, that the Zilber-Pink conjecture may be reduced to two statements, one of an arithmetic nature, the other a functional transcendence statement. In general, the former statement remains conjectural in both settings. In the abelian setting, the functional transcendence statement follows from a theorem of Ax [3], while in the modular setting a proof of it has been announced recently by Pila-Tsimerman [34]. Both statements are generalisations of statements which have been used to establish cases of the André-Oort conjecture, and this aspect of our work is in the spirit of Ullmo [47].

The arithmetic hypothesis, which we formulate here, is the "Large Galois Orbit" hypothesis (LGO) and asserts that, for fixed $V \subseteq X$, certain ("optimal") isolated intersection points of V with a special subvariety T have a "large" Galois orbit over a fixed finitely generated field of definition for V, expressed in terms of a suitable complexity measure of T. Special subvarieties in our settings are described in Section 2 for abelian varieties and Section 3.2 for $Y(1)^n$ and LGO is formulated in Section 8.

In the context of the André-Oort Conjecture, the Generalized Riemann Hypothesis (GRH) suffices to guarantee LGO (see [46, 48]). However, it is not clear to the authors if a variant of the Riemann Hypothesis leads to large Galois orbits for isolated points in $V \cap T$ if $\dim T \geq 1$. Indeed, in the Shimura setting, there seems to be no Galois-theoretic description of isolated points in $V \cap T$ which is rooted in class field theory. On the other hand, suitable bounds have been established unconditionally for André-Oort in several cases, and perhaps LGO will be found accessible without assuming GRH.

Associated with X is a certain transcendental uniformization $\pi : U \to X$. The functional transcendence hypothesis is the "Weak Complex Ax" hypothesis (WCA) and is a weak form of an analogue for π of "Ax-Schanuel" for cartesian powers of the exponential function. The latter result, due to Ax [2], affirms Schanuel's Conjecture (see [24, p. 30]) for differential fields. WCA is formulated in Section 5.

In the modular case $X = Y(1)^n$ our result is the following. A very special case of it was established unconditionally by us in [21].

Theorem 1.2. – *If LGO and WCA hold for $Y(1)^n$ then the Zilber-Pink Conjecture holds for subvarieties of $Y(1)^n$ defined over $\mathbb{C}$. Moreover, if WCA holds for $Y(1)^n$ and LGO holds with $K = \mathbb{Q}$, then the Zilber-Pink Conjecture holds for subvarieties of $Y(1)^n$ defined over $\overline{\mathbb{Q}}$.*

In the case that X is an abelian variety, we establish the same result in Section 9. However, as mentioned above, in this case WCA is known, and LGO can be established in the case that V is one-dimensional when X and V are defined over $\overline{\mathbb{Q}}$. This allows us to prove the above unconditional result for curves.

All current approaches towards Theorem 1.1 require a height upper bound on the set of points in question. Like many of the papers cited above we use Rémond's height bound [43] which relies on his generalization of Vojta's height inequality.

In contrast to previous approaches we do not rely on delicate Dobrowolski-type [12, 13, 40] or Bogomolov-type [18] height lower bounds to pass from bounded height to finiteness. These height lower bounds are expected (but not known) to generalize to arbitrary abelian varieties. Instead we will use a variation of the strategy originally devised by Zannier to reprove the Manin-Mumford Conjecture [36] for abelian varieties. This approach relied on the Pila-Wilkie point counting result in o-minimal structures. We will still require a height lower bound. However, the robust nature of the method allows us to use Masser's general bound [27] which predates the sophisticated and essentially best-possible results of Ratazzi and Carrizosa that require the ambient abelian variety to have complex multiplication.

In her recent Ph.D. thesis, Capuano [11] counted rational points on suitable definable subsets of a Grassmanian to obtain finiteness results on unlikely intersections with curves in the algebraic torus.

In the next theorem we collect several partial results in the abelian setting for subvarieties of arbitrary dimension.

Theorem 1.3. – *Let $V \subseteq X$ be a subvariety of an abelian variety, both defined over a number field K. Let us also fix an ample, symmetric line bundle on X and its associated Néron-Tate height $\hat{h}$.*

(i) *If $S \geq 0$ then*

$$\left\{P \in V(\overline{K}) \cap X^{[1+\dim V]};\ \hat{h}(P) \leq S\right\}$$

is contained in a finite union of proper algebraic subgroups of X.