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A COMPLETE CHARACTERIZATION OF CONNECTED LIE GROUPS WITH THE APPROXIMATION PROPERTY

BY Uffe HAAGERUP, Søren KNUDBY AND Tim DE LAAT

ABSTRACT. – We give a complete characterization of connected Lie groups with the Approximation Property for groups (AP). To this end, we introduce a strengthening of property (T), that we call property (T*), which is a natural obstruction to the AP. In order to define property (T*), we first prove that for every locally compact group G, there exists a unique left invariant mean m on the space $M_0A(G)$ of completely bounded Fourier multipliers of G. A locally compact group G is said to have property (T*) if this mean m is a weak* continuous functional. After proving that the groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$, and $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ have property (T*), we address the question which connected Lie groups have the AP. A technical problem that arises when considering this question from the point of view of the AP is that the semisimple part of the global Levi decomposition of a connected Lie group need not be closed. Because of an important permanence property of property (T*), this problem vanishes. It follows that a connected Lie group has the AP if and only if all simple factors in the semisimple part of its Levi decomposition have real rank 0 or 1. Finally, we are able to establish property (T*) for all connected simple higher rank Lie groups with finite center.

RÉSUMÉ. – Nous donnons une caractérisation complète des groupes de Lie connexes ayant la propriété d'approximation (AP) pour des groupes. À cette fin, nous introduisons un renforcement de la propriété (T), que nous appelons propriété (T*) et qui est une obstruction naturelle à AP. Dans le but de définir la propriété (T*), nous montrons d'abord que pour tout groupe localement compact G, l'espace $M_0A(G)$ des multiplicateurs complètement bornés de G admet une unique moyenne invariante à gauche m. Un groupe localement compact G a la propriété (T*) si m est une forme continue pour la topologie *-faible. Après avoir démontré que les groupes $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$ et $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ ont la propriété (T*), nous étudions la question de savoir lesquels parmi les groupes de Lie connexes ont l'AP. Il se pose alors le problème technique que la partie semi-simple de la décomposition de Levi globale d'un groupe de Lie connexe n'est pas toujours fermée. Grâce à une importante propriété de stabilité de la propriété (T*), ce problème disparaît. Il s'en suit qu'un groupe de Lie connexe a l'AP si et seulement

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si tous les facteurs simples de la partie semi-simple de sa décomposition de Levi ont un rang réel 0 ou 1. Enfin, nous démontrons que tous les groupes de Lie simples connexes de rang ≥ 2 et de centre fini ont la propriété (T*).

1. Introduction

The main aim of this article is to provide a complete characterization of connected Lie groups with the Approximation Property for groups (AP). It continues and relies on the work of Lafforgue and de la Salle [26] and the work of the first named and the third named author [20, 21].

A locally compact group G has the AP if there is a net (φ_α) in the Fourier algebra $A(G)$ of G such that $\varphi_\alpha \to 1$ in the weak* topology on $M_0A(G)$ (see Section 2 for details). The AP, which was introduced by the first named author and Kraus in [19], is the natural analogue for groups of the Banach Space Approximation Property (BSAP) of Grothendieck (see [18]). To see this, recall first that Banach spaces have a natural noncommutative analogue, namely, operator spaces. An operator space is a closed linear subspace E of the bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (see [13, 31]). For the class of operator spaces, which contains the class of C^*-algebras, a well-known analogue of the BSAP is known, namely, the operator space approximation property (OAP). The first named author and Kraus proved that a discrete group Γ has the AP if and only if its reduced C^*-algebra $C^*_\lambda(\Gamma)$ has the OAP.

The AP is strictly weaker than the well-known properties amenability and weak amenability. It has not been considered as much as these properties, or the Haagerup property for that matter, probably because until recently, the only examples of groups without the AP followed from the theoretical fact that every discrete group with the AP is exact, as established by the first named author and Kraus. However, in 2010, Lafforgue and de la Salle provided the first concrete examples of groups without the AP, namely, $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$ and lattices in these groups [26]. In fact, they also proved that $\mathrm{SL}(n,F)$ (with $n \geq 3$) and its lattices do not have the AP for any non-Archimedean local field F. Their result on real Lie groups was extended by the first named and the third named author of this article, first to connected simple Lie groups with real rank at least 2 and finite center [20], by proving that $\mathrm{Sp}(2,\mathbb{R})$ does not satisfy the AP, and then to all connected simple Lie groups with real rank at least 2 [21], by also considering the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. Indeed, any connected simple Lie group with real rank at least 2 contains a closed subgroup locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$, which, together with the known permanence properties of the AP, implies the general result. From this, it follows that a connected semisimple Lie group G that is isomorphic to a direct product $S_1 \times \cdots \times S_n$ of connected simple Lie groups has the AP if and only if all these S_i's have real rank 0 or 1 (see [21]).

It is straightforward to characterize the AP for more general classes of groups, by using the known permanence properties of the AP, but we are not able to provide a complete characterization of connected Lie groups with the AP in this way. However, by using a natural obstruction to the AP that we introduce in this article, it turns out that we can. This obstruction, which we call property (T*), is in fact a strengthening of property (T).

Property (T), as introduced by Kazhdan in [24], is a rigidity property for groups that has lead to striking results in several areas of mathematics (see [2]). Recall that a locally compact group has property (T) if its trivial representation is isolated in the unitary dual of the group equipped with the Fell topology.

As mentioned above, property (T*) forms a natural obstruction to the AP, in the sense that a locally compact group having both the AP and property (T*) is necessarily compact. Note that in the same way, property (T) is an obstruction to the Haagerup property.

For a locally compact group G, let $M_0A(G)$ denote the space of completely bounded Fourier multipliers of G (see Section 2). In order to define property (T*), we first prove the following result.

Theorem A. – Let G be a locally compact group. Then the space $M_0A(G)$ carries a unique left invariant mean m. This mean is also right invariant.

It is known that $M_0A(G)$ is a subspace of the space $W(G)$ of weakly almost periodic functions on G, which is known to have a unique left invariant mean. It follows that the mean on $M_0A(G)$ is the restriction to $M_0A(G)$ of the mean on $W(G)$. It is known from [16, Chapitre III] that also the Fourier-Stieltjes algebra $B(G)$ (see Section 2) of a locally compact group G has a unique left invariant mean.

As mentioned before, $M_0A(G)$ carries a weak* topology (see Section 2.3).

Definition 1.1. – A locally compact group G is said to have property (T*) if the unique left invariant mean m on $M_0A(G)$ is a weak* continuous functional.

It is easy to see that compact groups have property (T*). Also, as mentioned before, any group satisfying both the AP and property (T*) is compact. Using a powerful result of Veech from [36] and the results of [20] and [21], we are able to prove the following result.

Theorem B. – The groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$, and the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$ have property (T*).

Property (T*) satisfies certain permanence properties. One of the essential ones for us is that whenever $\pi\colon H \to G$ is a continuous homomorphism between locally compact groups with dense image and H has property (T*), then G has property (T*) (see Proposition 5.9). Using Theorem B and this permanence property, we are able to prove the following theorem, which gives a complete characterization of connected Lie groups with the AP. The statement of the theorem uses the Levi decomposition of connected Lie groups (see Section 2.6), asserting that any connected Lie group G admits a decomposition $G = RS$, where R is a solvable closed normal subgroup of G and S is a semisimple subgroup of G. The semisimple Lie group S is locally isomorphic to a direct product of connected simple factors.

Theorem C. – Let G be a connected Lie group, let $G = RS$ be a Levi decomposition, and suppose that S is locally isomorphic to the direct product $S_1 \times \cdots \times S_n$ of connected simple factors. Then the following are equivalent:

(i) the group G has the AP,
(ii) the group S has the AP,
(iii) the groups S_i, where $i = 1, \dots, n$, have the AP,