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WILD RAMIFICATION DETERMINES THE CHARACTERISTIC CYCLE

BY TAKESHI SAITO AND YURI YATAGAWA

ABSTRACT. – Constructible complexes have the same characteristic cycle if they have the same wild ramification, even if the characteristics of the coefficients fields are different.

RÉSUMÉ. – Des complexes constructibles ont le même cycle caractéristique s'ils ont la même ramification sauvage, même si les caracteristiques des corps de coefficients sont différentes.

The characteristic cycle $CC\,\mathcal{F}$ of a constructible complex $\mathcal{F}$ on a smooth variety X over a perfect field k is defined in [8], as a cycle on the cotangent bundle T^*X supported on the singular support $SS\,\mathcal{F}$ defined by Beilinson in [1]. The characteristic cycle is characterized by the Milnor formula recalled in Theorem 1.3 computing the total dimension of the space of vanishing cycles.

We show that constructible complexes have the same characteristic cycle if they have the *same wild ramification*. This terminology will be defined in Definition 5.1 in the text.

THEOREM 0.1. – *Let X be a smooth scheme over a perfect field k and let Λ and Λ' be finite fields of characteristic invertible in k. Let $\mathcal{F}$ and $\mathcal{F}'$ be constructible complexes of Λ-modules and of Λ'-modules on X respectively. If $\mathcal{F}$ and $\mathcal{F}'$ have the* same wild ramification, *we have*

$$CC\,\mathcal{F} = CC\,\mathcal{F}'. \tag{0.1}$$

A special case where $\Lambda = \Lambda'$ is proved in the thesis of the second named author [11, Theorem 7.25]. Theorem 0.1 is a refinement of and is deduced from the following equality of Euler characteristic.

PROPOSITION 0.2 (cf. [5, Théorème 2.1]). – *Let X be a separated scheme of finite type over an algebraically closed field k and let Λ and Λ' be finite fields of characteristic invertible in k. Let $\mathcal{F}$ and $\mathcal{F}'$ be constructible complexes of Λ-modules and of Λ'-modules on X respectively. If $\mathcal{F}$ and $\mathcal{F}'$ have the* same wild ramification, *we have*

$$\chi_c(X, \mathcal{F}) = \chi_c(X, \mathcal{F}'). \tag{0.2}$$

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A special case where $\Lambda = \Lambda'$ is proved in [5, Théorème 2.1].

To deduce Theorem 0.1 from Proposition 0.2, we take a morphism to a curve and use the Grothendieck-Ogg-Shafarevich formula to recover the total dimension of the space of vanishing cycles appearing in the characterization of characteristic cycle from the Euler-Poincaré characteristic.

We briefly recall the definition and properties of singular support and characteristic cycle in Section 1. As preliminaries of proof of Theorem 0.1, we prove the existence of a good pencil in Section 2. We show that the characteristic cycle of a sheaf is determined by the Euler-Poincaré characteristics of its pull-backs using the existence of a good pencil in Section 3. Finally, we prove Theorem 0.1 after defining the condition for constructible complexes to have the same wild ramification in Section 4.

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1. Characteristic cycle

We briefly recall the definition of characteristic cycle. We refer to [8] for more detail. For a smooth scheme X over a field k, let $T^*X = \operatorname{Spec} S^\bullet \Omega_X^{1\vee}$ be the cotangent bundle of X and let $T^*_X X$ denote the zero section. A morphism $f\colon X \to Y$ of smooth schemes over k induces a linear mapping $df\colon X \times_Y T^*Y \to T^*X$ of vector bundles on X. We say that a closed subset C of a vector bundle is *conical* if C is stable under the action by the multiplicative group.

Definition 1.1 ([1, 1.2]). – Let X be a smooth scheme over a field k and let $C \subset T^*X$ be a closed conical subset.

1. Let $h\colon W \to X$ be a morphism of smooth schemes over k. We say that h is *C-transversal* if we have

$$dh^{-1}(T^*_W W) \cap h^*C \subset W \times_X T^*_X X,$$

where $h^*C = W \times_X C$.

For a C-transversal morphism $h\colon W \to X$, we define a closed conical subset $h^\circ C \subset T^*W$ to be the image of $h^*C \subset W \times_X T^*X$ by the morphism $dh\colon W \times_X T^*X \to T^*W$.

2. Let $f\colon X \to Y$ be a morphism of smooth schemes over k. We say that f is *C-transversal* if we have

$$df^{-1}(C) \subset X \times_Y T^*_Y Y.$$

3. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k. We say that the pair (h, f) is *C-transversal* if h is C-transversal and if f is $h^\circ C$-transversal.

4. Let $j\colon U \to X$ be an étale morphism, $f\colon U \to Y$ a morphism over k to a smooth curve over k, and $u \in U$ a closed point. We say that u is an *isolated characteristic point* with respect to C if the pair (j, f) is not C-transversal and its restriction to $U - \{u\}$ is C-transversal.

Let Λ be a finite field of characteristic ℓ invertible in k. We say that a complex $\mathcal{F}$ of étale sheaves of Λ-modules on X is *constructible* if the cohomology sheaf $\mathcal{H}^q(\mathcal{F})$ is constructible for every q and if $\mathcal{H}^q(\mathcal{F}) = 0$ except finitely many q.

Definition 1.2 ([1, 1.3]). – Let X be a smooth scheme over a field k and let Λ be a finite field of characteristic ℓ invertible in k. Let $\mathcal{F}$ be a constructible complex of Λ-modules on X.

1. Let $C \subset T^*X$ be a closed conical subset. We say that $\mathcal{F}$ is *micro-supported* on C if for every C-transversal pair (h, f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k, the morphism f is locally acyclic relatively to $h^*\mathcal{F}$.

2. The *singular support* $SS\mathcal{F}$ of $\mathcal{F}$ is the smallest closed conical subset C of T^*X on which $\mathcal{F}$ is micro-supported.

By [1, Theorem 1.3], the singular support exists for every constructible complex of Λ-modules. Further, if X is equidimensional of dimension n, the singular support is equidimensional of dimension n.

Theorem 1.3 (Milnor formula, [8, Theorem 5.9, Theorem 5.18])

Let X be a smooth scheme equidimensional of dimension n over a perfect field k and let Λ be a finite field of characteristic ℓ invertible in k. Let $\mathcal{F}$ be a constructible complex of Λ-modules on X and $C \subset T^*X$ a closed conical subset. Assume that $\mathcal{F}$ is micro-supported on C and that every irreducible components C_a of $C = \bigcup_a C_a$ is of dimension n.

Then, there exists a unique $\mathbf{Z}$-linear combination $A = \sum_a m_a C_a$ satisfying the following condition: Let (j, f) be the pair of an étale morphism $j\colon U \to X$ and a morphism $f\colon U \to Y$ over k to a smooth curve over k. Let $u \in U$ be a closed point such that u is at most an isolated characteristic point of f with respect to C. Then we have

$$-\operatorname{dimtot}\phi_u(j^*\mathcal{F}, f) = (j^*A, df)_{T^*U,u}. \tag{1.1}$$

Further A is independent of C on which $\mathcal{F}$ is micro-supported.

In (1.1), the left hand side denotes the minus of the total dimension of the stalk $\phi_u(j^*\mathcal{F}, f)$ at u of the complex of vanishing cycles. The total dimension dimtot is defined as the sum of the dimension and the Swan conductor. The right hand side denotes the intersection number supported on the fiber of u of the pull-back j^*A with the section df defined to be the pull-back of dt for a local coordinate t of Y at $f(u)$.

Definition 1.4 ([8, Definition 5.10]). – Let X be a smooth scheme over a perfect field k and let Λ be a finite field of characteristic ℓ invertible in k. Let $\mathcal{F}$ be a constructible complex of Λ-modules on X. We define the *characteristic cycle* $CC\mathcal{F}$ of $\mathcal{F}$ to be $A = \sum_a m_a C_a$ in Theorem 1.3.

For $\mathbf{Z}_\ell$-coefficient or $\mathbf{Q}_\ell$-coefficient, the characteristic cycle is defined by taking the reduction modulo ℓ. Theorem 1.3 implies the following additivity of characteristic cycles. For a distinguished triangle $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ of constructible complexes of Λ-modules, we have

$$CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''. \tag{1.2}$$