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ON STABILITY AND HYPERBOLICITY FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

BY Pierre BERGER AND Romain DUJARDIN

ABSTRACT. – Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of polynomial automorphisms of $\mathbb{C}^2$. Following previous work of Dujardin and Lyubich, we say that such a family is weakly stable if saddle periodic orbits do not bifurcate. It is an open question whether this property is equivalent to structural stability on the Julia set J^* (that is, the closure of the set of saddle periodic points).

In this paper we introduce a notion of regular point for a polynomial automorphism, inspired by Pesin theory, and prove that in a weakly stable family, the set of regular points moves holomorphically. It follows that a weakly stable family is probabilistically structurally stable, in a very strong sense. Another consequence of these techniques is that weak stability preserves uniform hyperbolicity on J^*.

RÉSUMÉ. – Soit $(f_\lambda)_{\lambda\in\Lambda}$ une famille holomorphe d'automorphismes polynomiaux de $\mathbb{C}^2$. En accord avec un travail précédent de Dujardin et Lyubich, nous disons qu'une telle famille est faiblement stable si ses points périodiques ne bifurquent pas. La question est ouverte de savoir si cette notion équivaut à celle de stabilité structurelle sur l'ensemble de Julia J^* (qui est par définition l'adhérence de l'ensemble des points périodiques selles).

Dans cet article nous introduisons une notion de point régulier pour un tel automorphisme, inspirée par la théorie de Pesin, et montrons que dans une famille faiblement stable, l'ensemble des points réguliers se déplace selon un mouvement holomorphe. Nous en déduisons qu'une famille faiblement stable est structurellement stable en un sens probabiliste. Une autre conséquence de cette étude est que la stabilité faible préserve l'hyperbolicité uniforme sur J^*.

1. Introduction

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of polynomial automorphisms of $\mathbb{C}^2$, with non-trivial dynamics [(1)], parameterized by a connected complex manifold Λ. A basic stability/bifurcation dichotomy in this setting was introduced by M. Lyubich and the second author in [10]. In that paper it was proved in particular that under a moderate

(1) A necessary and sufficient condition for this is that the *dynamical degree* $d = \lim(\deg(f^n))^{1/n}$ satisfies $d \geq 2$, see § 2 for more details.

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dissipativity assumption [(2)], weakly stable parameters together with parameters exhibiting a homoclinic tangency form a dense subset of Λ. This confirms in this setting a (weak version of a) well-known conjecture of Palis. The notion of stability into consideration here is the following: a family is said to be *weakly stable* if periodic orbits do not bifurcate. Specifically, this means that the eigenvalues of the differential do not cross the unit circle.

In one-dimensional holomorphic dynamics, this seemingly weak notion of stability actually leads to the usual one of structural stability (on the Julia set or on the whole sphere) thanks to the theory of *holomorphic motions* developed independently by Mañé, Sad and Sullivan and Lyubich [18, 15, 16].

As it is well-known, the basic theory of holomorphic motions breaks down in dimension 2, and a corresponding notion of *branched holomorphic motion* (where collisions are allowed), was designed in [10]. To be more specific, let J^* be the closure of the set of saddle periodic orbits. It was shown by Bedford, Lyubich and Smillie that J^* contains all homoclinic and heteroclinic intersections of saddle points, and conversely, if p is any saddle point, then $W^s(p)\cap W^u(p)$ is dense in J^*. It was proved in [10] that if $(f_\lambda)_{\lambda\in\Lambda}$ is weakly stable, then there is an equivariant branched holomorphic motion of J^*, that is unbranched over the set of periodic points and homoclinic (resp. heteroclinic) intersections. This means that such points have a unique holomorphic continuation in the family, and furthermore, this continuation cannot collide with other points in J^* (see below § 2.1 for more details). The underlying idea is that the motion is unbranched on sets satisfying a local (uniform) expansivity property.

Still, it remains an open question whether a weakly stable family is structurally stable on J^*. A weaker version of this question, which is natural in view of the above analysis, is whether the unbranching property holds generically with respect to hyperbolic invariant probability measures.

The first main goal in this paper is to answer this second question. We introduce a notion of *regular point*, simply defined as follows: $p \in J^*$ is regular if there exists a sequence of saddle points $(p_n)_{n\geq 1}$ converging to p such that $W^u_{\mathrm{loc}}(p_n)$ and $W^s_{\mathrm{loc}}(p_n)$ are of size uniformly bounded from below as $n \to \infty$ and do not asymptotically coincide (see below § 4 for the formal definition, and § 3.2 for the notion of the local size of a manifold). The set $\mathcal{R}$ of regular points is invariant and dense in J^* since it contains saddle points and homoclinic intersections. More interestingly, Katok's closing lemma [14] implies that $\mathcal{R}$ is of full mass relative to any hyperbolic invariant probability measure (see below Proposition 4.4 for a precise statement and a sketch of proof). Observe however that our definition of regular point makes no reference to any invariant measure. Notice also that in our context, thanks to the Ruelle inequality, any invariant measure with positive entropy is hyperbolic.

Our first main result is the following.

THEOREM A. – *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, that is weakly stable.*

Then the set of regular points moves holomorphically and without collisions. More precisely, for every $\lambda \in \Lambda$, every regular point of f_λ admits a unique continuation under the branched

[(2)] That is, the complex Jacobian $\mathrm{Jac}(f)$ satisfies $|\mathrm{Jac}(f)| < d^{-2}$.

motion of J^*_λ, which remains regular in the whole family. In particular, the restrictions $f_\lambda|_{\mathcal{R}_\lambda}$ are topologically conjugate.

The meaning of the word "substantial" will be explained in § 2.1 below; it will be enough for the moment to note that any dissipative family is substantial by definition. By "topologically conjugate" we mean that there exists a homeomorphism $h : \mathcal{R}_\lambda \to \mathcal{R}_{\lambda'}$ such that $h \circ f_\lambda = f_{\lambda'} \circ h$ in restriction to $\mathcal{R}_\lambda$.

Let us say that a polynomial automorphism f is *probabilistically structurally stable* (in some given family (f_λ)) if for every f' sufficiently close to f, there exists a set $\mathcal{R}_f$ (resp. $\mathcal{R}_{f'}$) which is of full measure with respect to any hyperbolic invariant probability measure for f (resp. f') together with a continuous conjugacy $\mathcal{R}_f \to \mathcal{R}_{f'}$.

Recall also from the work of Friedland and Milnor [12] that every dynamically non-trivial polynomial automorphism is conjugate to a composition of Hénon mappings.

Theorem A and Corollary 4.5 in [10] suggest a description of structurally stable dissipative mappings as those far from displaying infinitely many sinks or far from displaying a homoclinic tangency. The above Theorem A enables us to go one step further in this direction.

Corollary B. – *Let f be a composition of Hénon mappings in $\mathbb{C}^2$. Then:*

- *f can be approximated in the space of polynomial automorphisms of degree d either by a probabilistically structurally stable map, or by one possessing infinitely many sinks or sources.*
- *If f is moderately dissipative and not probabilistically structurally stable, then f is a limit of automorphisms displaying homoclinic tangencies.*

The main step of the proof of Theorem A consists in studying how the size of local stable and unstable manifolds of a given saddle point varies in a weakly stable family. More precisely, assume that for some $\lambda_0 \in \Lambda$, $p(\lambda_0)$ is a saddle point such that $W^s(p(\lambda_0))$ has bounded geometry at scale r_0 at $p(\lambda_0)$. Since (f_λ) is weakly stable, $p(\lambda_0)$ persists as a saddle point $p(\lambda)$ for $\lambda \in \Lambda$. In § 3, we give estimates on the geometry of $W^s_{\mathrm{loc}}(p(\lambda))$ *which depend only on* r_0, based on the extension properties of the branched holomorphic motion of J^* along unstable manifolds devised in [10].

These estimates are used to control the geometry of the local "center stable manifold" of $\{(\lambda, p(\lambda)),\ \lambda \in \Lambda\}$, which is of codimension 1 in $\Lambda \times \mathbb{C}^2$. With this codimension 1 subset at hand, we can prevent collisions between the motion of points in J^* using classical tools from complex geometry (like the persistence of proper intersections and the Hurwitz Theorem).

We actually prove a more general version of Theorem A, which involves only regularity in one of the stable or the unstable directions (see Theorem 4.10 below). One motivation for this is that in the dissipative setting it is possible in certain situations to take advantage of dissipativity to obtain a good control on the geometry of stable manifolds (see Example 4.9).

If f is uniformly hyperbolic on J^*, then it is well known that $f|_{J^*}$ is structurally stable. In particular, if $(f_\lambda)_{\lambda\in\Lambda}$ is any family of polynomial automorphisms, and $\lambda_0 \in \Lambda$ is such that f_{λ_0} is uniformly hyperbolic on $J^*_{\lambda_0}$, then (f_λ) is (weakly) stable in some neighborhood of λ_0. Thus, λ_0 belongs to a *hyperbolic component* in Λ, where this uniform hyperbolicity is