ISSN 0012-9593

quatrième série - tome 50 fascicule 3 mai-juin 2017

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Tarifs

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 50, 2017, p. 755 à 786

MODERATE DEVIATIONS FOR THE RANGE OF A TRANSIENT RANDOM WALK: PATH CONCENTRATION

BY Amine ASSELAH AND Bruno SCHAPIRA

ABSTRACT. – We study downward deviations of the boundary of the range of a transient walk on the Euclidean lattice. We describe the optimal strategy adopted by the walk in order to shrink the boundary of its range. The technics we develop apply equally well to the range, and provide pathwise statements for the *Swiss cheese* picture of Bolthausen, van den Berg and den Hollander [7].

RÉSUMÉ. – Nous étudions les déviations qui réduisent la frontière du support d'une marche transiente sur le réseau euclidien. Nous décrivons en particulier une stratégie optimale pour réduire la frontière du support. Les techniques employées s'appliquent aussi bien au volume du support lui-même, et fournissent des énoncés mathématiques qui illustrent l'image du « fromage suisse » de Bolthausen, van den Berg et den Hollander.

1. Introduction

In this paper we study downward deviations of the boundary of the range of a simple random walk $(S_n,\ n \in \mathbb{N})$ on $\mathbb{Z}^d$, with $d \geq 3$. The range at time n, denoted $\mathcal{R}_n$, is the set of visited sites $\{S_0, \dots, S_n\}$, and its boundary, denoted $\partial \mathcal{R}_n$, is the set of sites of $\mathcal{R}_n$ with at least one neighbor outside $\mathcal{R}_n$. Our previous study [3] focused on the typical behavior of the boundary of the range, whereas this work is devoted to downward deviations and applications to a hydrophobic polymer model. The zest of the paper is about describing the optimal strategy adopted in order to shrink the boundary of the range, and our approach shed some light on the shape of the walk realizing such a deviation. In [3], we emphasized the ways in which, for a transient walk, the range and its boundary share a similar nature. Thus, even though the boundary of the range is our primary interest, we mention at the outset that the technics we develop apply equally well to the range. Since this last issue has been the focus of many celebrated works, let us describe first the state of the art there.

 doi:10.24033/asens.2331
ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE

1.0.0.1. *Deviations of the range.–* A pioneering large deviation study of Donsker and Varadhan [11] establishes asymptotics for downward deviations of the volume of the Wiener sausage $t \mapsto W^a(t)$, that is the Lebesgue measure of an a-neighborhood of the standard Brownian motion. The main result of [11] establishes, in any dimension and for any $\beta > 0$, the following asymptotics

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}[\exp\big(-\beta W^a(t)\big)] = \frac{d+2}{2}\,\beta(\frac{2\lambda_D}{d\beta})^{\frac{d}{d+2}}, \tag{1.1}$$

where λ_D is the first eigenvalue of the Laplacian with Dirichlet condition on the boundary of a sphere of volume one. The asymptotics (1.1), obtained in the random walk setting in [12], correspond to downward deviation of the volume of the range $\{|\mathcal{R}_n| \le f(n)\}$ where $|\mathcal{R}_n|$ denotes the volume of $\mathcal{R}_n$ and $f(n)$ is of order $n^{\frac{d}{d+2}}$. They suggest that during time n a random walk is localized in a ball of radius $(n/\beta)^{\frac{1}{d+2}}$ filled without holes. Bolthausen [9] and Sznitman [23], with different technics, extended the result of [11] to cover downward deviations corresponding to $f(n) = n^{1-\delta}$ for any $\delta > 0$. A consequence of their analysis is that for $0 < \gamma \le 2$

$$\lim_{t\to\infty} t^{-\frac{d+\gamma-2}{d+\gamma}} \log \mathbb{E}[\exp\big(-\beta\, t^{-\frac{2-\gamma}{d+\gamma}} W^a(t)\big)] = -\frac{d+2}{2}\beta(\frac{2\lambda_D}{d\beta})^{\frac{d}{d+2}}. \tag{1.2}$$

Then, three deep studies dealt with the trajectory conditioned on realizing a large deviation by Sznitman [24], Bolthausen [10] and Povel [21]. The case $\gamma = 0$ in (1.2) was recognized as *critical* by Bolthausen [9], and indeed a different behavior was later proved to hold [7]. The series of papers on downward deviations culminated in a paper of Bolthausen, van den Berg and den Hollander [7] which covers the *critical regime* $\{|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|] \le -\varepsilon n\}$. The latter contribution offers a precise Large Deviation Principle, but no pathwise statement characterizing the most likely scenario. The present paper is a step towards filling this gap and providing answers to their motto *How a Wiener sausage turns into a Swiss cheese*? Let us quote their mathematical results. In dimension $d \ge 3$, $\mathbb{E}[W^a(t)]$ grows linearly and the limit of $\frac{1}{t}\mathbb{E}[W^a(t)]$ is denoted κ_a (the Newtonian capacity of a ball of radius a). It is proved in [7] that for any $0 < \varepsilon < 1$

$$\lim_{t\to\infty} \frac{1}{t^{\frac{d-2}{d}}} \log \mathbb{P}\big[W^a(t) - \mathbb{E}[W^a(t)] \le -\varepsilon\kappa_a t\big] = -I_a(\varepsilon), \tag{1.3}$$

where

$$I_a(\varepsilon) = \frac{1}{2\kappa_a^{2/d}} \inf\{\|\nabla f\|_2 :\ f \in H^1(\mathbb{R}^d), \|f\|_2 = 1,\ \int_{\mathbb{R}^d} \big(1-\exp(-f^2(x))\big)dx\ \le 1-\varepsilon\}. \tag{1.4}$$

A similar result for simple random walks is obtained in Phetpradap's thesis [20]: κ_a becomes the non-return probability say κ_d, and the factor $1/2\kappa_a^{2/d}$ in (1.4) becomes $1/2d\kappa_d^{2/d}$. When $d = 3$ or $d = 4$, the minimizers of (1.4) are strictly positive on $\mathbb{R}^d$, and decrease in the radial component. This is interpreted as saying that Wiener sausage *"looks like a Swiss cheese" with random holes whose sizes are of order 1 and whose density varies on scale* $t^{1/d}$. On the other hand, when $d \ge 5$, and when the parameter ε in (1.3) is small, there is no minimizer for the variational problem (1.4), suggesting that *the optimal strategy is time-inhomogeneous.*

1.0.0.2. *Boundary of the range.–* The boundary of the range, in spite of not receiving much attention, enters naturally into the modeling of *hydrophobic polymers*. Indeed, a *polymer* is a succession of monomers centered at the positions of the walk (and thus covering $\mathcal{R}_n$), the complement of the range is occupied by the aqueous solvent, and being *hydrophobic* means that the monomers try to hide from it. A natural model is then the following polymer measure depending on two parameters: its length n, and its inverse temperature β,

$$d\widetilde{\mathbb{Q}}_n^\beta = \frac{1}{\widetilde{Z}_n(\beta)} \exp(-\beta|\partial\mathcal{R}_n|)\, d\mathbb{P}_n,$$

where $\mathbb{P}_n$ denotes the law of the simple random walk up to time n and $\widetilde{Z}_n(\beta)$, the partition function, is a normalizing factor. Biology suggests that as one tunes β, for a fixed polymer length, a phase transition appears. The recent results of Berestycki and Yadin [6] treat an asymptotic regime of length going to infinity, and suggest that for any positive β a long enough polymer, that is under $\widetilde{\mathbb{Q}}_n^\beta$, is localized in a ball of radius ρ_n with ρ_n^{d+1} of order n. Thus, to capture the insight from Biology, we rather scale β with $n^{2/d}$, when n is taken to infinity. We therefore consider

$$d\mathbb{Q}_n^\beta = \frac{1}{Z_n(\beta)} \exp\Big(-\frac{\beta}{n^{2/d}}\big(|\partial\mathcal{R}_n| - \mathbb{E}[|\partial\mathcal{R}_n|]\big)\Big)\, d\mathbb{P}_n. \tag{1.5}$$

The centering of $|\partial\mathcal{R}_n|$ is a matter of taste, but the scaling of β by $n^{2/d}$ is crucial, and corresponds to *a critical regime* for the boundary of the range reminiscent of (1.2) for $\gamma = 0$. Indeed, understanding the polymer measure is linked with analyzing the scenarii responsible for shrinking the boundary of the range on the scale of its mean. However, before tackling deviations, let us recall some typical behavior of the boundary of the range. Okada [19] has proved a law of large numbers in dimension $d \geq 3$, and when dimension is two, he proved that

$$\frac{\pi^2}{2} \leq \lim_{n\to\infty} \frac{\mathbb{E}[|\partial\mathcal{R}_n|]}{n/(\log n)^2} \leq 2\pi^2.$$

Note that Benjamini, Kozma, Yadin and Yehudayoff [5] in their study of the entropy of the range have obtained the correct order of magnitude for $\mathbb{E}[|\partial\mathcal{R}_n|]$ in $d = 2$, and have linked the entropy of the range to the size of its boundary.

In addition, a central limit theorem for the boundary of the range was proved in [3] in dimension $d \geq 4$. When dimension is three, the variance is expected to grow like $n \log n$, and only an upper bound of the right order is known [3]. We henceforth focus on the ways in which a random walk reduces the boundary of its range.

1.0.0.3. *Capacity of the range.–* A key object used to probe the shape of the random walk is the *capacity of its range*. We first define it, and then state our result. For $\Lambda \subset \mathbb{Z}^d$, let H_Λ^+ be the time needed by the walk to return to Λ. The capacity of Λ, denoted cap(Λ), is

$$\mathrm{cap}(\Lambda) = \sum_{x\in\Lambda} \mathbb{P}_x\big[H_\Lambda^+ = \infty\big]. \tag{1.6}$$

Let us recall one of its basic property. There exists a positive constant c_{cap}, such that for all finite subset $\Lambda \subset \mathbb{Z}^d$

$$c_{\mathrm{cap}}\, |\Lambda|^{1-\frac{2}{d}} \leq \mathrm{cap}(\Lambda) \leq |\Lambda|. \tag{1.7}$$