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Alexandr ANDONI & Assaf NAOR & Ofer NEIMAN

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

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SNOWFLAKE UNIVERSALITY OF WASSERSTEIN SPACES

BY ALEXANDR ANDONI, ASSAF NAOR AND OFER NEIMAN

ABSTRACT. – For $p \in (1, \infty)$ let $\mathscr{P}_p(\mathbb{R}^3)$ denote the metric space of all p-integrable Borel probability measures on $\mathbb{R}^3$, equipped with the Wasserstein p metric W_p. We prove that for every $\varepsilon > 0$, every $\theta \in (0, 1/p]$ and every finite metric space (X, d_X), the metric space (X, d_X^θ) embeds into $\mathscr{P}_p(\mathbb{R}^3)$ with distortion at most $1 + \varepsilon$. We show that this is sharp when $p \in (1, 2]$ in the sense that the exponent $1/p$ cannot be replaced by any larger number. In fact, for arbitrarily large $n \in \mathbb{N}$ there exists an n-point metric space (X_n, d_n) such that for every $\alpha \in (1/p, 1]$ any embedding of the metric space (X_n, d_n^α) into $\mathscr{P}_p(\mathbb{R}^3)$ incurs distortion that is at least a constant multiple of $(\log n)^{\alpha - 1/p}$. These statements establish that there exists an Alexandrov space of nonnegative curvature, namely $\mathscr{P}_2(\mathbb{R}^3)$, with respect to which there does not exist a sequence of bounded degree expander graphs. It also follows that $\mathscr{P}_2(\mathbb{R}^3)$ does not admit a uniform, coarse, or quasisymmetric embedding into any Banach space of nontrivial type. Links to several longstanding open questions in metric geometry are discussed, including the characterization of subsets of Alexandrov spaces, existence of expanders, the universality problem for $\mathscr{P}_2(\mathbb{R}^k)$, and the metric cotype dichotomy problem.

RÉSUMÉ. – Pour $p \in (1, \infty)$ notons $\mathscr{P}_p(\mathbb{R}^3)$ l'espace métrique des mesures de probabilité p-intégrables sur $\mathbb{R}^3$, muni de la p-métrique de Wasserstein W_p. Nous montrons que pour tout $\varepsilon > 0$, tout $\theta \in (0, 1/p]$ et tout espace métrique fini (X, d_X), l'espace métrique (X, d_X^θ) se plonge dans $\mathscr{P}_p(\mathbb{R}^3)$ avec distortion au plus $1 + \varepsilon$. Nous montrons que cela est optimal quand $p \in (1, 2]$ au sens où l'exposant $1/p$ ne peut pas être augmenté. En fait pour $n \in \mathbb{N}$ assez grand il existe un espace métrique à n points (X_n, d_n) tel que pour tout $\alpha \in (1/p, 1]$ tout plongement de l'espace métrique (X_n, d_n^α) dans $\mathscr{P}_p(\mathbb{R}^3)$ a une distortion au moins égale à un multiple par une constante de $(\log n)^{\alpha - 1/p}$. Ces résultats impliquent qu'il existe un espace d'Alexandrov de courbure positive, à savoir $\mathscr{P}_2(\mathbb{R}^3)$, vis- à-vis duquel il n'existe pas de suite de graphes expanseurs de degré borné. Il en résulte aussi que $\mathscr{P}_2(\mathbb{R}^3)$ n'admet pas de plongement uniforme, grossier ou quasisymétrique dans un espace de Banach de type non trivial. Nous discutons le lien avec plusieurs questions ouvertes depuis longtemps en géométrie des espaces métriques, dont la caractérisation des sous-ensembles des espaces d'Alexandrov, l'existence d'expandeurs, le problème d'universalité pour $\mathscr{P}_2(\mathbb{R}^k)$, et le problème de dichotomie pour le cotype métrique.

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1. Introduction

We shall start by quickly recalling basic notation and terminology from the theory of transportation cost metrics; all the necessary background can be found in [96]. For a complete separable metric space (X, d_X) and $p \in (0,\infty)$, let $\mathscr{P}_p(X)$ denote the space of all Borel probability measures μ on X satisfying

$$\int_X d_X(x,x_0)^p \mathrm{d}\mu(x) < \infty$$

for some (hence all) $x_0 \in X$. A coupling of a pair of Borel probability measures (μ,ν) on X is a Borel probability measure π on $X \times X$ such that $\mu(A) = \pi(A\times X)$ and $\nu(A) = \pi(X \times A)$ for every Borel measurable $A \subseteq X$. The set of couplings of (μ,ν) is denoted $\Pi(\mu,\nu)$. The Wasserstein p distance between $\mu,\nu \in \mathscr{P}_p(X)$ is defined to be

$$\mathsf{W}_p(\mu,\nu) \stackrel{\mathrm{def}}{=} \inf_{\pi\in\Pi(\mu,\nu)} \left(\iint_{X\times X} d_X(x,y)^p \mathrm{d}\pi(x,y)\right)^{\frac{1}{p}}.$$

W_p is a metric on $\mathscr{P}_p(x)$ whenever $p \geqslant 1$. The metric space $(\mathscr{P}_p(X), \mathsf{W}_p)$ is called the Wasserstein p space over (X, d_X). Unless stated otherwise, in the ensuing discussion whenever we refer to the metric space $\mathscr{P}_p(X)$ it will be understood that $\mathscr{P}_p(X)$ is equipped with the metric W_p.

1.1. Bi-Lipschitz Embeddings

Suppose that (X, d_X) and (Y, d_Y) are metric spaces and that $D \in [1,\infty]$. A mapping $f : X \to Y$ is said to have distortion at most D if there exists $s \in (0,\infty)$ such that every $x, y \in X$ satisfy $s d_X(x,y) \leqslant d_Y(f(x), f(y)) \leqslant D s d_X(x,y)$. The infimum over those $D \in [1,\infty]$ for which this holds true is called the distortion of f and is denoted $\mathbf{dist}(f)$. If there exists a mapping $f : X \to Y$ with distortion at most D then we say that (X, d_X) embeds with distortion D into (Y, d_Y). The infimum of $\mathbf{dist}(f)$ over all $f : X \to Y$ is denoted $c_{(Y,d_Y)}(X, d_X)$, or $c_Y(X)$ if the metrics are clear from the context.

1.2. Snowflake universality

Below, unless stated otherwise, $\mathbb{R}^n$ will be endowed with the standard Euclidean metric. Here we show that $\mathscr{P}_p(\mathbb{R}^3)$ exhibits the following universality phenomenon.

THEOREM 1. – *If $p \in (1,\infty)$ then for every finite metric space (X, d_X) we have*

$$c_{(\mathscr{P}_p(\mathbb{R}^3),\mathsf{W}_p)}\left(X, d_X^{\frac{1}{p}}\right) = 1.$$

For a metric space (X, d_X) and $\theta \in (0,1]$, the metric space (X, d_X^θ) is commonly called the θ-*snowflake* of (X, d_X); see e.g., [21]. Thus Theorem 1 asserts that the θ-snowflake of any finite metric space (X, d_X) embeds with distortion $1+\varepsilon$ into $\mathscr{P}_p(\mathbb{R}^3)$ for every $\varepsilon \in (0,\infty)$ and $\theta \in (0, 1/p]$ (formally, Theorem 1 makes this assertion when $\theta = 1/p$, but for general $\theta \in (0,1/p]$ one can then apply Theorem 1 to the metric space $(X, d_X^{\theta p})$ to deduce the seemingly more general statement).

Theorem 2 below implies that Theorem 1 is sharp if $p \in (1,2]$, and yields a nontrivial, though probably non-sharp, restriction on the embeddability of snowflakes into $\mathscr{P}_p(\mathbb{R}^3)$ also for $p \in (2,\infty)$.

THEOREM 2. – *For arbitrarily large $n \in \mathbb{N}$ there exists an n-point metric space (X_n, d_{X_n}) such that for every $\alpha \in (0,1]$ we have*

$$c_{(\mathscr{P}_p(\mathbb{R}^3),\mathsf{W}_p)}(X_n, d_{X_n}^\alpha) \gtrsim \begin{cases} (\log n)^{\alpha-\frac{1}{p}} & \text{if } p \in (1,2], \\ (\log n)^{\alpha+\frac{1}{p}-1} & \text{if } p \in (2,\infty). \end{cases}$$

Here, and in what follows, we use standard asymptotic notation, i.e., for $a,b \in [0,\infty)$ the notation $a \gtrsim b$ (respectively $a \lesssim b$) stands for $a \geqslant cb$ (respectively $a \leqslant cb$) for some universal constant $c \in (0,\infty)$. The notation $a \asymp b$ stands for $(a \lesssim b) \wedge (b \lesssim a)$. If we need to allow the implicit constant to depend on parameters we indicate this by subscripts, thus $a \lesssim_p b$ stands for $a \leqslant c_p b$ where c_p is allowed to depend only on p, and similarly for the notations $\gtrsim_p$ and $\asymp_p$.

We conjecture that when $p \in (2,\infty)$ the lower bound in Theorem (2) could be improved to

$$c_{(\mathscr{P}_p(\mathbb{R}^3),\mathsf{W}_p)}(X_n, d_{X_n}^\alpha) \gtrsim_p (\log n)^{\alpha-\frac{1}{2}},$$

and, correspondingly, that the conclusion of Theorem 1 could be improved to state that if $p \in (2,\infty)$ then $c_{(\mathscr{P}_p(\mathbb{R}^3),\mathsf{W}_p)}\big(X, \sqrt{d_X}\big) \lesssim_p 1$ for every finite metric space (X, d_X); see Question 23 below.

There are several motivations for our investigations that led to Theorem 1 and Theorem 2. Notably, we are inspired by a longstanding open question of Bourgain [13], as well as fundamental questions on the geometry of Alexandrov spaces. We shall now explain these links.

1.3. Alexandrov geometry

We need to briefly present some standard background on metric spaces that are either nonnegatively curved or nonpositively curved in the sense of Alexandrov; the relevant background can be found in e.g., [18, 15]. Let (X, d_X) be a complete geodesic metric space. Recall that $w \in X$ is called a metric midpoint of $x, y \in X$ if $d_X(x,w) = d_X(y,w) = d_X(x,y)/2$. The metric space (X, d_X) is said to be an Alexandrov space of nonnegative curvature if for every $x, y, z \in X$ and every metric midpoint w of x, y,

$$d_X(x,y)^2 + 4d_X(z,w)^2 \geqslant 2d_X(x,z)^2 + 2d_X(y,z)^2. \tag{1}$$

Correspondingly, the metric space (X, d_X) is said to be an Alexandrov space of nonpositive curvature, or a Hadamard space, if for every $x, y, z \in X$ and every metric midpoint w of x, y,

$$d_X(x,y)^2 + 4d_X(z,w)^2 \leqslant 2d_X(x,z)^2 + 2d_X(y,z)^2. \tag{2}$$

If (X, d_X) is a Hilbert space then, by the parallelogram identity, the inequalities (1) and (2) hold true as equalities (with $w = (x+y)/2$). So, (1) and (2) are both natural relaxations of a stringent Hilbertian identity (both relaxations have far-reaching implications). A complete Riemannian manifold is an Alexandrov space of nonnegative curvature if and only if its