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QUASI-PROJECTIVITY OF THE MODULI SPACE OF SMOOTH KÄHLER-EINSTEIN FANO MANIFOLDS

BY CHI LI, XIAOWEI WANG AND CHENYANG XU

ABSTRACT. – In this paper, we prove that there is a canonical continuous Hermitian metric on the CM line bundle over the proper moduli space $\overline{\mathcal{M}}$ of smoothable Kähler-Einstein Fano varieties. The Chern curvature of this Hermitian metric is the Weil-Petersson current, which exists as a closed positive (1,1)-current on $\overline{\mathcal{M}}$ and extends the canonical Weil-Petersson current on the moduli space $\mathcal{M}$ of smooth Kähler-Einstein Fano manifolds. As a consequence, we show that the CM line bundle is nef and big on $\overline{\mathcal{M}}$ and its restriction on $\mathcal{M}$ is ample.

RÉSUMÉ. – Dans cet article, nous montrons qu'il existe une métrique hermitienne continue et canonique sur le fibré en droites CM au-dessus de l'espace de modules $\overline{\mathcal{M}}$ des variétés de Kähler-Einstein régularisables. La courbure de Chern de cette métrique hermitienne est le courant de Weil-Petersson, qui existe en tant que (1,1)-courant fermé positif sur $\overline{\mathcal{M}}$, et étend le courant canonique de Weil-Petersson défini sur l'espace de modules $\mathcal{M}$ des variétés de Kähler-Einstein Fano régulières. Nous montrons aussi, en guise d'application de notre résultat, que le fibré des lignes CM est nef et big sur $\overline{\mathcal{M}}$, et que sa restriction à $\mathcal{M}$ est ample.

1. Introduction

The study of moduli spaces of polarized varieties is a fundamental topic in algebraic geometry. The most classical case is the moduli space of Riemann surfaces of genus ≥ 2, whose compactification is a Deligne-Mumford stack admitting a projective coarse moduli space. People have been trying to generalize this picture to higher dimensions, leading to the development of KSBA compactification of moduli space of canonically polarized varieties (see [37]). In [66], Viehweg proved a deep result that the moduli space of polarized manifolds with nef canonical line bundles is quasi-projective. Building on the fundamental work of [35] and the development of Minimal Model Program, it is proved in [29] that the KSBA compactification is projective.

On the other hand, there are negative results concerning the projectivity of moduli spaces. Kollár [36] showed that the moduli space of polarized manifolds may not be quasi-projective. In particular, he proved that any toric variety can be a moduli space of polarized uniruled

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manifolds. The quasi-projectivity of moduli is still open for polarized manifolds that are not uniruled and whose canonical line bundles are not nef.

Differential geometric methods have also played important roles in studying moduli spaces of complex manifolds. For examples, the moduli space $\mathscr{M}_g$ has been studied using Teichmüller spaces equipped with Weil-Petersson metrics; the moduli spaces of Calabi-Yau manifolds and its Weil-Petersson metrics were studied by Tian and Todorov. Moreover, Tian's results in [58] imply that there is a Hermitian line bundle on the moduli space of Calabi-Yau manifolds whose curvature form is the Weil-Petersson metric. Fujiki-Schumacher later [28] considered the more general case of moduli space of Kähler manifolds admitting constant scalar curvature Kähler (cscK) metrics. They proved that the natural Weil-Petersson metric is always Kähler by interpreting it as the Chern curvature of a determinant line bundle equipped with a Quillen metric. To achieve this, they applied the study of determinant line bundles and the Quillen metrics by Bismut-Gillet-Soulé [11]. As a consequence, it was proved in [28] that any compact subvariety in the moduli space of cscK manifolds with discrete automorphisms is projective. However, all the cases considered above require the fibration to be smooth, which is not the case in general. In [60] Tian studied similar determinant line bundles in a singular setting and introduced the notion of CM ($\mathbb{Q}$-)line bundle (see Definition 4.2 and Remark 4.3), which will be denoted by λ_{CM} from now on in this paper.

It follows from Kollár's negative result that, for uniruled manifolds extra constraints must be imposed in order for the moduli space to be quasi-projective/projective. However, Tian's study of CM line bundle and Fujiki-Schumacher's results suggest that the moduli space of manifolds admitting canonical metrics could be quasi-projective. In this paper, we confirm this speculation for the moduli space of Fano Kähler-Einstein (KE) manifolds, which was first conjectured by Tian in [60].

Fano manifolds in dimension 2 are called del Pezzo surfaces. It was proved in [59] that a smooth del Pezzo surface admits a Kähler-Einstein metric if and only if its automorphism is reductive. Recently, based on the study of degenerations of smooth Kähler-Einstein del Pezzo surfaces in [59], proper moduli spaces of smoothable Kähler-Einstein del Pezzo varieties were constructed in [47]. Moreover, it was shown in [47] that these proper moduli spaces are actually projective except possibly for the case of del Pezzo surfaces of degree 1.

The higher dimensional generalization of the results in [59] and [47] was made possible thanks to the celebrated solutions to the Yau-Tian-Donaldson conjecture ([14], [15], [16], [64]). The moduli space of higher dimensional smooth Kähler-Einstein Fano manifolds, denoted by $\mathscr{M}$ from now on, was studied in [62], [24], [46]. More recently, a proper algebraic compactification $\overline{\mathscr{M}}$ of $\mathscr{M}$ was constructed in [41] (see also [45]). It is further believed that $\overline{\mathscr{M}}$ should be projective (see [47], [41], [45]). This paper is a step towards establishing this. The main technical result of this paper is the following descent and extension result.

THEOREM 1.1. – *The* CM *line bundle λ_{CM} descends to a $\mathbb{Q}$-line bundle Λ_{CM} on the proper moduli space $\overline{\mathscr{M}}$. There is a canonically defined continuous Hermitian metric h_{DP} on Λ_{CM} whose curvature form is a positive current ω_{WP} on $\overline{\mathscr{M}}$ which extends the canonical Weil-Petersson current $\omega^{\circ}_{\mathrm{WP}}$ on $\mathscr{M}$.*

We remark that the descending of CM line bundle has been expected once the existence of a *good moduli* in the sense of [2] is verified, see e.g., [47, Section 6.2]. In this paper we will give a detailed account of this fact based on our construction of $\overline{\mathscr{M}}$ in [41] and Kempf's descending criterion (see [2, Theorem 10.3] and [26, Theorem 2.3]). The main challenge remaining is proving its positivity.

Note that in Theorem 1.1 although we use the notion of current on singular complex spaces defined in Definition 2.6, $\omega_{\mathrm{WP}}^{\circ}$ is actually a smooth Kähler-metric on a dense open set $\mathscr{M}'$ of $\mathscr{M}$ (see Section 4.1.1 and the proof of Theorem 1.2 in Section 6). So equivalently, we can say that ω_{WP} extends the canonical smooth Kähler metric $\omega_{\mathrm{WP}}^{\circ}|_{\mathscr{M}'}$ on $\mathscr{M}'$.

With $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ at hand (or equivalently, Weil-Petersson current ω_{WP} with a controlled behavior), we can apply a quasi-projective criterion as Theorem 6.1 to get the following result.

THEOREM 1.2. – Λ_{CM} *is nef and big over* $\overline{\mathscr{M}}$. *Moreover, for the normalization morphism* $n : \overline{\mathscr{M}}^{\mathrm{n}} \to \overline{\mathscr{M}}$ *which induces an isomorphism over* $\mathscr{M}$, *the rational map* $\Phi_{|n^*(m\Lambda_{\mathrm{CM}})|}$ *associated to the complete linear system* $|n^*(m\Lambda_{\mathrm{CM}})|$ *embeds* $\mathscr{M}$ *into* $\mathbb{P}^{N_m-1}$ *for* $m \gg 1$ *with* $N_m = \dim H^0(\overline{\mathscr{M}}^{\mathrm{n}}, n^*(m\Lambda_{\mathrm{CM}}))$. *In particular,* $\mathscr{M}$ *is quasi-projective.*

In some sense, the quasi-projectivity of $\mathscr{M}$ in Theorem 1.2 could be seen as a consequence of Tian's partial C^0-estimates recently established in the fundamental works of Donaldson-Sun [25] and Tian [63]. Actually such kind of implication was stated without proof in [60, end of Section 8] which used the notion of CM stability (introduced in [60]). However, because of the subtlety pointed out by Kollár [36], it is still not clear to us how to deduce the quasi-projectivity directly using CM stability. Nevertheless, if we only look at the open locus of $\mathscr{M}$ which parametrizes Kähler-Einstein Fano manifolds with finite automorphism groups, then [20] and [46] have already shown that it is quasi-projective as we know the Fano manifolds it parametrizes are all asymptotically Chow stable by [20]. On the other hand, if we drop the finite automorphism assumption, there exists a Kähler-Einstein Fano manifold which is asymptotically Chow unstable (see [48]).

Our proof of Theorem 1.1, which heavily depends on the recent development in the theory of Kähler-Einstein metrics on Fano varieties, is also inspired by the work of Schumacher-Tsuji [55] and Schumacher [53]. In [53], Schumacher re-proved the quasi-projectivity of $\mathscr{M}^-$ which is the moduli space of canonically polarized manifolds by using some compactification of $\mathscr{M}^-$ and the extension of Weil-Petersson metric. Our argument uses a similar approach. First, by applying the theory of Deligne pairings, for any smooth variey S together with a flat family of Kähler-Einstein Fano varieties $\mathscr{X} \to S$ containing an open dense $S^\circ \subset S$ such that the fibers of $\mathscr{X}|_{S^\circ} \to S^\circ$ are all Kähler-Einstein Fano manifolds, we can construct a Hermitian metric h_{DP} on the CM line bundle $\lambda_{\mathrm{CM}} \to S$ whose restriction to S° is the classical Weil-Petersson metric. Second, the partial-C^0 estimate established in [25, 63] together with an extension of continuity results in [39] allow us to show that this metric is indeed *continuous* whose curvature form can be extended to a positive current on S. Third, by using the local GIT description of the canonical compactification $\overline{\mathscr{M}}$ constructed in [41] and the fact that $\overline{\mathscr{M}}$ is a *good moduli* in the sense of [2] (see Section 5), CM line bundle λ_{CM} together with the metric h_{DP} can be descended to an Hermitian line bundle $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ on $\overline{\mathscr{M}}$, whose curvature form is exactly the Weil-Petersson current we want. The descending