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A FLOER FUNDAMENTAL GROUP

BY JEAN-FRANÇOIS BARRAUD

ABSTRACT. – The main purpose of this paper is to provide a description of the fundamental group of a symplectic manifold in terms of Floer theoretic objects. As an application, we show that when counted with a suitable notion of multiplicity, non degenerate Hamiltonian diffeomorphisms have enough fixed points to generate the fundamental group.

RÉSUMÉ. – L'objet de cet article est de donner une description du groupe fondamental d'une variété symplectique en terme d'objets de la théorie de Floer. À titre d'application, on montre que tous les difféomorphismes hamiltoniens non dégénérés ont, si on les compte avec une notion convenable de multiplicité, suffisamment de points fixes pour engendrer le groupe fondamental.

1. Introduction

1.1. Presentation of the results

In many ways, the topology of a space influences its geometry, and this is particularly true in symplectic geometry. Having a symplectic interpretation of a topological invariant is a good tool to explore this relationship. The celebrated Floer Homology ([7, 6]) is of course a strong illustration of this phenomenon. Introduced to prove the homological version of the Arnold conjecture ([1]), it quickly became one of the most powerful tools in symplectic geometry.

However, all the techniques derived from the original Floer construction are homological, or at least chain complex based in nature. The notion of cobordism (among moduli spaces) is even at the root of the original ideas of M. Gromov [9] of using pseudo-holomorphic curves to derive invariants in symplectic geometry. The use of local coefficients in Floer complexes allows Floer theory to involve some homotopical invariants, but purely homotopical tools are still missing, and it is the goal of this paper to provide a Floer theoretic interpretation of the fundamental group.

All the objects this construction is based on are still classical Floer theoretic objects, but the essential non Abelian phenomena that make the difference between the fundamental

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group and the first homology group are caught by a deeper use of 1-dimensional moduli spaces, and the use of "augmentations".

More precisely, let (M, ω) be a connected closed monotone symplectic manifold and choose an Hamiltonian function H on M, a possibly time dependent almost complex structure J compatible with ω, and a point $\star$ in M to serve as the base point. Recall the Floer trajectories in this setting are (finite energy) maps $u : \mathbb{R} \times S^1 \to M$ satisfying the Floer equation

$$\frac{\partial u}{\partial s}(s,t) + J_t(u(s,t))\frac{\partial u}{\partial t}(s,t) = J_t(u(s,t))X_{H_t}(u(s,t)),$$

where X_H is the Hamiltonian vector field associated to H.

Using a cutoff function χ to turn off the non homogeneous Hamiltonian term on the positive end of the tube (resp. on both ends but preserving it on an annulus of varying modulus) allows to define moduli spaces denoted by $\mathcal{M}(x, \emptyset)$ (resp. $\mathcal{M}(\star, \emptyset)$), which are Floer counterparts of Morse unstable manifolds (see the comments after Definition 2.4). It is a classical result of Floer theory that for a generic set of auxiliary data $(H, J, \star, \chi)$, all these moduli spaces are smooth finite dimensional manifolds.

Similarly to the Morse setting where a loop can be seen as a concatenation of paths associated to unstable manifolds of index 1 critical points, we use the components of the above 1-dimensional moduli spaces to define a notion of Floer loop (see Definition 2.8). These loops come naturally with concatenation and cancelation relations for which they form a group $\mathcal{L}(H, \star)$. The main statement of the paper is then the following theorem:

THEOREM 1.1. – *There is a natural evaluation map that induces a surjective group homomorphism* $\mathcal{L}(H, J, \star) \twoheadrightarrow \pi_1(M, \star)$.

A description of the relations is also given, but, although they obviously only depend on H, J, $\star$, and χ, we resort to an auxiliary Morse function to get a finite presentation for them (see Section 4). Nevertheless, we produce explicit relations such that the generated normal subgroup $\mathcal{R}(H, \star)$ satisfies the following statement:

THEOREM 1.2. – *The evaluation map induces a group isomorphism*

$$\mathcal{L}(H, \star)/\mathcal{R}(H, \star) \xrightarrow{\sim} \pi_1(M, \star) .$$

Notice the construction is presented here in the absolute setting, i.e., Hamiltonian fixed points problem, but also makes sense in the relative one, i.e., intersections of a Lagrangian sub-manifold with its deformations under Hamiltonian isotopies problem. Although the latter can be expected to hold the most interesting applications, we choose to focus on the former for the sake of simplicity and to highlight better the main ideas: the generalization to the latter entails exactly the same issues as for the homology and involves no new idea.

Finally, the construction also makes sense in the stable Morse setting (i.e., study of Morse functions that are quadratic at infinity on $M \times \mathbb{R}^N$). Although the corresponding results have their own interest and would deserve a separate discussion, they will only be quickly sketched without proofs in the last section of this paper (see Section 6), rather as an illustration and a simplified finite dimensional model of the Floer setting.

A natural outcome of this construction is an estimate on the number of fixed points of Hamiltonian diffeomorphisms, but not in usual way, since a notion of multiplicity has to be introduced. Indeed, rather than the critical points themselves, the relevant objects required to build loops are their unstable manifolds (called "steps" in the sequel), and while to one critical point corresponds exactly one unstable manifold in the Morse setting, Floer counterparts of unstable manifolds may have several components, which have all to be taken into account.

Counting the number $\nu_J(x)$ (resp. $\nu_J(\star)$) of steps through a given Conley-Zehnder index $1-n$ fixed point x (resp. $\star$) defines a notion of multiplicity for these points (that depends on the almost complex structure, see Definition 2.10 for more details). We then have the following theorem:

THEOREM 1.3. – *Let $\rho(\pi_1(M))$ denote the minimal number of generators of the fundamental group. Then:*

$$\nu_J(\star) + \sum_{|y|=1} \nu_J(y) \geq \rho(\pi_1(M)), \tag{1}$$

where the sum runs over the contractible 1*-periodic orbits, or more precisely over the homotopy classes of cappings of such orbits with Conley-Zehnder index* $1-n$.

REMARK 1. – The number $\nu_J(\star)$ is a sum of contributions of index $-n$ fixed points (see Definition 2.10), so that inequality (1) can be interpreted as a lower bound for the number of particular Floer configurations associated to fixed points with Conley-Zehnder index $-n$ and $1-n$.

REMARK 2. – This statement should be compared to its Morse analog, namely that for any Morse function $f : M \to \mathbb{R}$, we have

$$\sharp\mathrm{Crit}_1(f) \geq \rho(\pi_1(M)), \tag{2}$$

where $\mathrm{Crit}_1(f)$ denotes the set of index 1 critical points.

As already mentioned, the construction, and hence the definition of the multiplicities makes sense in the stable Morse and a fortiori Morse settings (moreover, we claim, without proof, that for C^2 small Morse functions, Morse and Floer moduli spaces can be identified like in [7], and that multiplicities coincide in this case). For an index 1 Morse critical point y, $\nu_J(y)$ is the number of components of its unstable manifold, and hence always evaluates to 1. Similarly, $\nu_J(\star)+1$ is the number of Morse trajectories through $\star$ and hence evaluates to 1. As a consequence, $\nu_J(\star)=0$, so that

$$\nu_J(\star) + \sum_{|y|=1} \nu_J(y) = \sharp\mathrm{Crit}_1(f),$$

and (1) appears as a generalization of (2) to the more general Floer setting.

REMARK 3. – There is no hope to avoid multiplicities in (1) as long as they result from a construction that also applies to the stable Morse setting, which is the case of ours.

Indeed, M. Damian showed in [4] that the stable Morse number (which is the minimal number of critical points of a Morse function which is quadratic at infinity on a product $M \times \mathbb{R}^N$) may be strictly smaller than the Morse number (which is the minimal