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SOME REMARKS ABOUT ALGEBRAIC INDEPENDENCE MEASURES IN HIGH DIMENSION

BY

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RÉSUMÉ. — Pour $\omega \in \mathbb{C}^n$ et $k \in \mathbb{N}$, $1 \leq k \leq n$, il est possible de définir deux mesures. La première est la *mesure d'indépendance algébrique*, qui donne une minoration de la grandeur en ω d'un idéal I de polynômes de codimension k. La deuxième est la *mesure d'approximation*, qui donne une minoration d'approximabilité des coordonnées de ω par des nombres algébriques sur une extension transcendante pure de $\mathbb{Q}$ de dimension $n-k$.

En une variable, ces mesures sont équivalentes entre elles. En plusieurs variables les relations qui les lient ne semblent pas optimales. Dans ce texte nous étudions des mesures d'indépendance algébrique pour "idéaux lisses" afin d'établir des relations plus précises avec les mesures d'approximation.

ABSTRACT. — For $\omega \in \mathbb{C}^n$ and $k \in \mathbb{N}$, $1 \leq k \leq n$, it is possible to define two different measures. Firstly, an *algebraic independence measure*, which gives a lower bound for the "smallness" at ω of a k-rank ideal I of polynomials; secondly an *approximation measure*, which provides a lower bound for the approximability of ω with n-tuples of complex numbers, algebraic over a pure transcendental extension of dimension $n-k$ over the rationals. If $n=1$ these measures are equivalent, but in higher dimensions the relations between them do not seem to be optimal. In this paper algebraic independence measures "for smooth ideal" are analized and a stronger relation between them and approximation measures is found.

0. Introduction

In a recent paper ([P3]) PHILIPPON considers some relations between algebraic independence measures and approximation measures.

Let us recall some definitions. For any proper homogeneous unmixed ideal $I \subset \mathbb{Z}[x_0, \dots, x_n]$, with $I \cap \mathbb{Z} = \{0\}$, we define $t(I)$ as the size of the Chow form F associated with I. The norm $\|I\|_\omega$ of I at $\omega \in \mathbb{C}^{n+1} - \{0\}$ is defined as $H(F(S^1\omega, \dots, S^{n-k+1}\omega))|\omega|^{-(n+1-k)\delta}$ where $k = \operatorname{rank} I$, δ is the degree of F in the first group of variables and $S^1, \dots, S^{n-k+1}$

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are skew-symmetric matrices in the new variables $s_{k\ell}^{(j)}$, $0 \leq k < \ell < n$, $j = 1, \dots, n-k+1$. Here $t(I)$ and $\|I\|_\omega$ play a role analogous to that of the size $t(Q)$ and the modulus $|Q(\omega)|$ for a polynomial $Q \in \mathbb{Z}[x]$.

Given $\omega \in \mathbb{C}^n$, we denote by ω also $(1, \omega) \in \mathcal{C}^{n+1} - \{0\}$. An algebraic independence measure in codimension k ($0 \leq k \leq n+1$) for ω is a function $\varphi_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$\|{}^h I\|_\omega \geq \exp\bigl(-\varphi_k(t(I))\bigr)$$

holds for any proper unmixed ideal $I \subset \mathbb{Z}[x_1, \dots, x_n]$ with $I \cap \mathbb{Z} = \{0\}$ and $\operatorname{rank} I = k$.

Let $\theta \in \mathbb{C}$ be algebraic over a purely transcendental extension $K = \mathbb{Q}(\alpha_1, \dots, \alpha_D)$. Let

$$f(y) = \sum_{h=0}^{\delta} P_h(\alpha_1, \dots, \alpha_D) y^h, \qquad P_h \in \mathbb{Z}[x_1, \dots, x_D],$$

be the irreducible polynomial of $\mathbb{Z}[x_1, \dots, x_D, y]$ such that $f(\alpha_1, \dots, \alpha_D, \theta) = 0$. The size of θ with respect to K is defined as

$$t_K(\theta) = \max_{0 \leq h \leq \delta} \bigl(t(P_h), \delta\bigr).$$

An approximation measure in dimension D ($0 \leq D \leq n+1$) for ω is a function $\gamma_D : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$|\theta - \omega| \geq \exp\Bigl(-\gamma_D\bigl(\max_{1 \leq i \leq n} (t_K(\theta_i))\bigr)\Bigr)$$

holds for any $\theta \in \mathbb{C}^n$ with algebraic coordinates over a pure transcendental extension K of $\mathbb{Q}$ with $\operatorname{tr}\deg_{\mathbb{Q}} K = D$.

PHILIPPON finds the following results :

THEOREM 0.1. — *Let $\omega \in \mathbb{C}^n$. There exist positive constants α_1, α_2, α_3, α_4 depending only on n and $|\omega|$, with the following properties. If φ_k is an algebraic independence measure in codimension k for ω, then the function*

$$\gamma_{n-k}(T) = \varphi_k(a_1 T^n) - a_2 T^n$$

is an approximation measure in dimension $n-k$ for ω. Conversely, let γ_D be an approximation measure in dimension D for ω. Then the function

$$\varphi_{n-D}(T) = T\bigl[\gamma_D(a_3 T) - a_4\bigr]$$

is an algebraic independence measure in codimension $n-d$ for ω.

If the approximation measures which can be derived from algebraic independence measures seem to be the best possible ones, in the opposite direction the result seems to be weaker. For example, if $n=1$, it is well-known (see [W, page 133] for instance) that $\varphi_1(T) = a_5 \max(\gamma_0(T), T^2)$ is an algebraic independence measure in codimension 1 for $\omega \in \mathbb{C}$ if γ_0 is any approximation measure in dimension 0 and a_5 is some constant depending only on $|\omega|$. The reason for this gap between dimension one and higher dimensions is in the "inversion theorem" (see [P2, lemma 2.7]) which is the main tool for the proof of the second part of THEOREM 0.1 :

THEOREM 0.2 ("Inversion theorem"). — *Let $\omega \in \mathbb{C}^n$. There exists a positive constant a_6 depending only on n and $|\omega|$ with the following properties. Let P be a proper prime ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ with $P \cap \mathbb{Z} = \{0\}$. Then there exists $\theta \in \mathbb{V}_{\mathbb{C}^n}(P) = \{x \in \mathbb{C}^n \mid f(x) = 0 \ \forall f \in P\}$ such that*

$$|\omega - \theta| \leq a_6 \, \|P\|_\omega^{1/t(P)}.$$

We observe that a stronger result is available in dimension one :

THEOREM 0.3. — *Let $\omega \in \mathbb{C}$. There exists a positive constant a_7 depending only on $|\omega|$ with the following properties. Let $Q \in \mathbb{Z}[x]$ be a prime polynomial. Then there exists a root α of Q such that*

$$|\omega - \alpha| \leq |Q(\omega)| \exp\bigl(a_7 t(Q)^2\bigr).$$

In this paper we extend THEOREM 0.3 to higher dimensions, under suitable "smoothness" conditions for the variety of zeros of P, obtaining a stronger version of THEOREM 0.2. These results yields the following improvements of THEOREM 0.1 :

THEOREM 0.4. — *Let $\omega \in \mathbb{C}^n$. There exists a positive constant a_8 depending only on n and $|\omega|$ with the following properties. Let φ_D be an approximation measure in dimension D for ω. Suppose that $\varphi_D(x)/x$ is an increasing function. Then for any proper unmixed ideal $I \in \mathbb{Z}[x_1, \ldots, x_n]$ with $I \cap \mathbb{Z} = \{0\}$ and $\operatorname{rank} I = n - D$ the following inequality*

$$\|{}^h I\|_\omega > \exp\Bigl(\max\Bigl(a_8 t(I)^{D+2}, \varphi_D\bigl(a_8 t(I)\bigr)\Bigr)\Bigr)$$

holds, provided that $\mathbb{V}_{\mathbb{P}(\mathbb{C}^n)}({}^h I)$ is a smooth variety. In other terms

$$\tilde{\psi}_{n-D}(T) = \max\bigl(a_8 T^{D+2}, \varphi_D(a_8 T)\bigr)$$

is an algebraic independence measure "for smooth ideals" in codimension $n - D$ *for* ω.

THEOREME 0.5. — *Let* $\omega \in \mathbb{C}^n$. *There exists a positive constant* a_9 *depending only on* n *and* $|\omega|$ *with the following properties. Let* φ_{D-1}, φ_D *be approximation measures in dimension* $D-1$, D *for* ω. *Suppose that* $\varphi_h(x)/x$ *is an increasing function for* $h = D-1$, D. *Then*

$$\psi_{n-D}(T) = \max\Big(a_9 T^{D+2}, \varphi_D(a_9 T),\, T^2 \varphi_{D-1}(a_9 T^2)\Big)$$

is an algebraic independence measure in codimension $n - D$ *for* ω.

1. Inversion of ideals

The following lemmas will be useful :

LEMMA 1.1. — *Let* P *be a prime homogeneous ideal of* $\mathbb{Z}[x_0, \dots, x_n]$ *with* $P \cap \mathbb{Z} = \{0\}$, $\operatorname{rank} P = k$. *Let* $f_1, \dots, f_m$ *be homogeneous polynomials of* $Z[x_0, \dots, x_n]$ *of total degree at most* δ. *Let* $r = \operatorname{rank}(P, f_1, \dots, f_m)$. *Then there exists homogeneous polynomials* $\lambda_{ij} \in \mathbb{Z}[x_0, \dots, x_n]$ *with* $(1 \le i \le r-k,\ 1 \le j \le m)$, $\deg \lambda_{ij} = \delta - \deg f_j$ *and* $H(\lambda_{ij}) \le \delta^{r-k} \deg P$ *such that the polynomials* $Q_1, \dots, Q_{r-k}$ *defined by*

$$Q_i = \lambda_{i1} f_1 + \cdots + \lambda_{im} f_m, \quad 1 \le i \le r-k,$$

are a "semi"-regular sequence in $\mathbb{Z}[x_0, \dots, x_n]/P$, *i.e. if* $I_0 = P$ *and we define for* $i = 1, \dots, r-k$ *the ideal* I_i *as the intersection of the isolated components of* (I_{i-1}, Q_i), *we have* $I_i : Q_{i+1} = I_i$ *for* $i = 1, \dots, r-k$.

The proof of this lemma is standard. See for instance [P2, lemma 1.9] or [M–W, lemma 2], although these authors take $P = 0$.

LEMMA 1.2. — *Let* P *be a homogeneous prime ideal of* $\mathbb{C}[x_0, \dots, x_n]$ *of rank* $n - d + 1$. *Let* $F \in \mathbb{C}[u_{ij}]_{i=1,\dots,d,\, j=0,\dots,n}$ *be a Chow form associated to* P. *Let us denote by* F_j, $j = 0, \dots, n$ *the derivation of* F *with respect to* u_{dj}. *For* $\omega \in \mathbb{C}^{n+1} - \{0\}$ *let us consider the homomorphism* $\sigma_\omega : \mathbb{C}[u_{ij}] \to \mathbb{C}[s^i_{k\ell}]_{0 \le k < \ell \le n,\, i=1,\dots,d}$ *given on each* u_i *by* $u_i \mapsto S^i \omega$ *where the* S^i *are skew-symmetric matrices in the variable* $s^i_{k\ell}$. *Let us assume*

$$\sigma_\omega F \equiv \sigma_\omega F_0 \equiv \cdots \equiv \sigma_\omega F_n \equiv 0.$$

Then ω *is a singular point of* $\mathbb{V} = \mathbb{V}_{\mathbf{P}(\mathbb{C}^n)}(P)$.

Proof. — Let us denote by δ the degree of F with respect to the variables u_i. Let us assume ω be a regular point of $\mathbb{V}$. We show that

$$\sigma_\omega F \equiv \sigma_\omega F_h \implies \omega_h = 0.$$