BULLETIN DE LA S. M. F.

FABIO PODESTÀ

A class of symmetric spaces

Bulletin de la S. M. F., tome 117, nº 3 (1989), p. 343-360

<http://www.numdam.org/item?id=BSMF_1989__117_3_343_0>

Bull. Soc. math. France,
117, 1989, p. 343–360.

A CLASS OF SYMMETRIC SPACES

BY

FABIO PODESTA (*)

RÉSUMÉ. — Le but de ce travail est d'étudier une variété C^∞ donnée d'une structure projective $\mathcal{P}$ et d'un système de symétries qui laissent $\mathcal{P}$ invariante. Avec une hypothèse supplémentaire d'homogénéité projective, l'auteur classifie tous ces espaces et donne une interprétation géométrique de l'inessentialité de la structure projective symétrique en utilisant des techniques de géométrie affine différentielle.

ABSTRACT. — The aim of this work is to study the situation of a C^∞-manifold endowed with a projective structure $\mathcal{P}$ and with a system of symmetries leaving $\mathcal{P}$ invariant. Under the additional hypothesis of projective homogeneity, the author classifies all such spaces and exhibits a geometrical interpretation for inessentiality of the symmetric projective structure using techniques of affine differential geometry.

Introduction

In a previous paper [8] the author has introduced the class of projectively symmetric spaces : let (M, ∇) be a connected C^∞ manifold with a linear torsion free connection ∇ on its tangent bundle; (M, ∇) is said to be *projectively symmetric* if for every point x of M there is an involutorial projective transformation of M fixing x and whose differential at s is $-\operatorname{Id}$. The assignement of the symmetry s_x at each point x of M is assumed to be not even continuous.

In this work the author gives necessary and sufficient conditions for a projectively symmetric and projectively homogeneous space to be inessential (*i.e.* projectively equivalent to an affine symmetric space, *see* paragraph 1). For complete Riemannian manifolds (M, g) of dimension n $(n \geq 3)$ that are projectively symmetric and projective homogeneous (it is shown with an example that projective homogeneity is not implied), the author proves that such spaces are either inessential or isometric to the sphere $S^n(r)$ of radius r or to the projective space $S^n(r)/\pm\operatorname{Id}$ with some choice of symmetries. Some interesting cases are considered, when (M, g)

(*) Texte reçu le 7 septembre 1988, révisé le 23 février 1989
F. PODESTA, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 PISA, Italie.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/1989/343/$ 5.00

is affinely homogeneous or analytic : under these hypotheses the author proves that (M, g) is either a Riemannian symmetric space (that is all projective symmetries are isometries) or (M, g) is isometric to the sphere $S^n(r)$ of radius r or to the projective space $S^n(r)/\pm \mathrm{Id}$ with some choice of symmetries. Finally the case of a smooth distribution of symmetries is considered and the previous results are rivisited from this point of view, exhibiting furthermore a geometrical interpretation for inessentiality.

The author wishes to express his hearthy thanks to Professor K. Nomizu for his encouragement and his valuable suggestions during the preparation of this paper.

1. Projectively symmetric spaces

Let M be a connected real C^∞ manifold whose tangent bundle TM is endowed with a linear torsion free connection ∇. We recall that a diffeomorphism s of M is said to be a projective transformation if s maps geodesics into geodesics when the parametrization is disregarded; equivalently s is projective if the pull back $s^*\nabla$ of the connection is projectively related to ∇, *i.e.* if there exists a global 1-form π on M such that

$$(1.1) \qquad s^*\nabla_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X \qquad \forall X, Y \in \mathcal{H}(M)$$

where $\mathcal{H}(M)$ denotes the Lie algebra of vector fields on M. If the form π vanishes identically on M, then s is said to be an affine transformation (see *e.g.* [1]). We remark here that all what follows could be made also in the case of a more general projective structure $\mathcal{P}$ defined on the manifold M, but we prefer to work with a projective equivalence class of globally defined linear connections and we shall denote with $[\nabla]$ the projective structure determined by the connection ∇.

Définition 1.1. — $(M, [\nabla])$ is said to be *projectively symmetric* if for every point x in M there exists a projective transformation s_x with the following properties :

(a) $s_x(x) = x$ and x is an isolated fixed point of s_x;
(b) s_x is involutorial;
(c) $ds_x|_x = -\mathrm{Id}$.

It is easy to see that conditions (a) and (b) imply (c). Moreover we recall that a projective transformation is determined if we fix its value at a point, its differential and its second jet at this point (see [3]), hence a symmetry at x in M is not uniquely determined in general by the conditions (a), (b) (and (c)).

Example 1. — Here is the simplest example of a projectively symmetric space. We consider S^n, the unit sphere in the euclidean space $\mathbb{R}^{n+1}$, endowed with the standard metric g. Then the group of projective transformations of (S^n, g) is naturally identified with $G = \mathrm{GL}(n+1)/\mathbb{R}^+$ under the action $\mu : S^n \times G \to S^n$ given by

$$\mu\big([g], x\big) = \frac{gx}{\|gx\|} \qquad \forall x \in S^n \tag{1.2}$$

where $[g]$ denotes the class of an element g of $\mathrm{GL}(n+1)$ in G and $\|\cdot\|$ denotes the euclidiean norm. It is easy to see that the action of G on S^n is effective and C^∞ (for more details, see [4]). We now fix $q = (1, 0, \ldots, 0)$ in S^n; then every projective symmetry s_q turns out be of the form

$$s_q(x) = \mu([A], x) \qquad \forall x \in S^n \tag{1.3}$$

where

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & -\mathrm{Id} \end{pmatrix} \tag{1.4}$$

with ${}^t\alpha \in \mathbb{R}^n$; moreover s_q is an affine transformation if and only if $\alpha = 0$. This simple example shows that no unique choice of s_q is possible.

Similar considerations hold also for the case of the real projective space $\mathbb{RP}^n$.

We now remark that every projective map carries geodesics, but does not preserve in general the affine parameter on the geodesics. However a projective transformation preserves the class of projective parameters (see [1]); nevertheless a projective involution with the properties (a), (b) (and (c)) does not carry necessarily a projective parameter p into $-p$, as it happens for the affine parameter in the classical theory of symmetric spaces. If we look at the previous example, we find that all the geodesics emanating from the point q and with projective parameter t are given by

$$\Gamma(t) = \big((1 + \|\xi\|^2 t^2)^{-1/2},\ t(1 + \|\xi\|^2 t^2)^{-1/2}\xi^i\big) \qquad t \in \mathbb{R},\ i = 1, \ldots, n$$

where $\xi \in \mathbb{R}^n \cong T_q S^n$. If we choose $\xi = (1, 0, \ldots, 0)$ and A is as in (1.2) with ${}^t\alpha = \xi$, then

$$\mu\big([A], \Gamma(t)\big) = \Gamma\big(-t(1 + \|\xi\|^2 t)^{-1}\big) \qquad \forall t > -\|\xi\|^{-2}.$$

This simple example shows how different the situation is from the affine case.

Remark 1. — It is clear how to construct examples of projectively but not affinely symmetric spaces; let M be a manifold with a linear torsionfree connection ∇ on its tangent bundle and suppose that (M, ∇) is affinely symmetric; fix a point q in M and denote by s_q the affine symmetry at q. Then if we choose a global 1-form π that is not s_q-invariant, it is enough to define a linear connection ∇^* via the formula

$$\nabla^*_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X \qquad \forall X, Y \in \mathcal{H}(M)$$

to obtain that M is projectively but not affinely symmetric with respect to the connection ∇^*. A general question is to find conditions under which, given a projectively symmetric space (M, ∇), there exists a projectively related connection ∇^* such that (M, ∇^*) is affinely symmetric; we shall call such spaces inessential projectively symmetric spaces (and essential otherwise). We now show that there is a choice of projective symmetries on the sphere S^n with respect to which S^n is essential : we denote with e the point ${}^t(1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ and choose as symmetry s the transformation induced by an element A of $\mathrm{GL}(n+1)$ as in (1.4) with a fixed $\alpha \in \mathbb{R}^n$, while we put as symmetry σ at the point $-e$ the transformation induced by an element B of $\mathrm{GL}(n+1)$ as in (1.4) with $\alpha' \neq \alpha$. The other symmetries are allowed to be chosen arbitrarily. We claim that S^n with this choice of symmetries is essential : indeed if it were inessential, then we would have that

$$s \circ \sigma = s_{s(-e)} \circ s = \sigma \circ s$$

and this is not the case because $\alpha' \neq \alpha$. Deeply related to this is the question whether projectively symmetric spaces are necessarily projectively homogeneous, since the classical techniques used in the theory of symmetric spaces fail in this case. So we are going to show that there exist Riemannian spaces that are projectively symmetric but not projectively homogeneous.

Example 2. — Example of a Riemannian manifold that is projectively symmetric but not projectively homogeneous.

We consider the real projective space $\mathbb{RP}^n$ ($n \geq 3$) and two distinct point p and q: we claim that the manifold $M = \mathbb{RP}^n \backslash \{p, q\}$ endowed with the restriction of the standard metric of $\mathbb{RP}^n$ is projectively symmetric but not projectively homogeneous. Indeed it is clear that a projective automorphism of M is the restriction to M of a projective transformation of $\mathbb{RP}^n$; so there is no projective transformation carrying a point $x \in M$ belonging to the line ℓ through p and q to a point y not belonging to ℓ. We have now to show that M is projectively symmetric. We fix a point $x \in M$ and consider the canonical projection $\pi : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$; pick