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INDUCED $\mathcal{D}$-MODULES AND DIFFERENTIAL COMPLEXES

BY

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RÉSUMÉ. — On introduit la notion de complexe différentiel et de $\mathcal{D}$-Module induit, on définit leur dual et image directe et on démontre la dualité pour un morphisme propre, ce qui implique la dualité de $\mathcal{D}$-Modules avec la compatibilité à celle de Verdier. On donne aussi une remarque sur la preuve de la correspondance de Riemann-Hilbert.

ABSTRACT. — We introduce the notion of differential complex and induced $\mathcal{D}$-Module, define their duals and direct images, and prove the duality for proper morphisms, which implies the duality of $\mathcal{D}$-Modules and its compatibility with the Verdier duality. A remark on the proof of the Riemann-Hilbert correspondence is also given.

Introduction

Let $f : X \to Y$ be a proper morphism of complex manifolds, or smooth algebraic varieties, and $M^\bullet \in D^b_{\mathrm{coh}}(\mathcal{D}_X)$ a bounded complex of $\mathcal{D}_X$-Modules with coherent cohomologies. Then we have the duality isomorphism (*cf.* also [B, Be, Sc1-2]) :

$$(0.1) \qquad f_*\mathbb{D}M^\bullet \xrightarrow{\sim} \mathbb{D}f_*M^\bullet \quad \text{in } D^b_{\mathrm{coh}}(\mathcal{D}_Y),$$

if $\mathcal{H}^jM^\bullet$ have good filtrations (locally on Y). Here $\mathbb{D}$ is the dual functor, and f_* is the direct image of $\mathcal{D}$-Modules. For simplicity, assume $X = \mathbb{P}^n$, $Y = \mathrm{pt}$, and M^j are direct sums of $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(p)$. Then we have the isomorphism (0.1) for each M^j by the Serre duality, but it is not completely trivial that the differentials on the both sides of (0.1) commute with the isomorphism, because we have to prove some relation between the duality isomorphism and the action of differential operators, *e. g.* the action of the global vector fields on $H^n(\mathbb{P}^n, \omega^n_{\mathbb{P}^n})$ is zero (this can be easily

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checked, if the distributions are used in the Serre duality). In this note we show that the duality isomorphism (0.1) is naturally defined in general by introducing the notion of induced $\mathcal{D}$-Module and differential complexe.

An *induced* $\mathcal{D}_X$-Module M is a right $\mathcal{D}_X$-Module isomorphic to $L \otimes_{\mathcal{O}_X} \mathcal{D}_X$ for an $\mathcal{O}_X$-Module L. Then we have $f_* M \simeq \mathbb{R} f_\bullet L \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ by definition, where $f_\bullet$ denotes the sheaf theoretic direct image, and f is always assumed to be proper. The dual $\mathbb{D}M$ of M is defined by

$$\mathbb{D}M = \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}\big(L, \omega_X[d_X]\big) \otimes_{\mathcal{O}_X} \mathcal{D}_X. \tag{0.2}$$

This definition coincides with the usual one,

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(L \otimes \mathcal{D}_X, \omega_X[d_X] \otimes D_X),$$

if L is coherent. Then the duality isomorphism (0.1) for such M is defined by

$$\begin{aligned} f_*\mathbb{D}M &= \mathbb{R}f_\bullet \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}\big(L, \omega_X[d_X]\big) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathbb{R}f_\bullet L, \mathbb{R}f_\bullet \omega_X[d_X]\big) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\xrightarrow{\mathrm{Tr}_f} \mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathbb{R}f_\bullet L, \omega_Y[d_Y]\big) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = \mathbb{D}f_* M \end{aligned} \tag{0.3}$$

using the analytic, or algebraic, trace morphism $\mathrm{Tr}_f : \mathbb{R}f_\bullet \omega_X[d_X] \to \omega_Y[d_Y]$, where $\omega_X = \Omega_X^{d_X}$ and $d_X = \dim X$. But we have to still impose some condition on the trace morphism for the compatibility of the morphism (0.3) with the differential of M, if M becomes a complex, and this condition is rather difficult to satisfy in the level of complex, *cf.* 3.14. This point can be further simplified by using the associated differential complexes.

For $\mathcal{O}_X$-Modules L, L' the *differential morphisms* of L to L' are the image of the *injective* morphism defined by $\otimes_{\mathcal{D}_X} \mathcal{O}_X$:

$$\mathcal{H}om_{\mathcal{D}_X}\big(L \otimes \mathcal{D}_X, L' \otimes \mathcal{D}_X\big) \longrightarrow \mathcal{H}om_{\mathbb{C}_X}(L, L') \tag{0.4}$$

where the image is denoted by $\mathcal{H}om_{\mathrm{Diff}}(L, L')$. This injectivity means that the morphisms of the induced Modules are completely recovered by the associated differential morphisms. The direct image of L is defined simply by the sheaf theoretic direct image, *i.e.* the differential morphisms are stable by the direct image. Then the above duality isomorphism (0.3) corresponds to

$$\begin{aligned} f_\bullet \mathcal{H}om^f_{\mathrm{Diff}}(L, \widetilde{K}_X) &\longrightarrow \mathcal{H}om^f_{\mathrm{Diff}}\big(\mathbb{R}f_\bullet L, f_\bullet \widetilde{K}_X\big) \\ &\xrightarrow{\mathrm{Tr}_f} \mathcal{H}om^f_{\mathrm{Diff}}\big(\mathbb{R}f_\bullet L, \widetilde{K}_Y\big) \end{aligned} \tag{0.5}$$

where $\widetilde{K}_X = DR(K_X)$ with $\omega_X[d_X] \to K_X$ an injective resolution as $\mathcal{D}_X$-Modules, and $\mathcal{H}om^f_{\mathrm{Diff}}$ denotes the subsheaf of $\mathcal{H}om_{\mathrm{Diff}}$ corresponding to

$$\bigcup_p \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes F_p\mathcal{D}_X) \hookrightarrow \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes \mathcal{D}_X)$$
$$= \mathcal{H}om_{\mathcal{D}_X}(L \otimes \mathcal{D}_X, L' \otimes \mathcal{D}_X).$$

Here the trace morphism $\mathrm{Tr}_f : f_\bullet \widetilde{K}_X \to \widetilde{K}_Y$ is defined to be compatible with the filtration F associated to the de Rham functor DR, and $\mathrm{Gr}^F_0 \mathrm{Tr}_f$ coincides with the above analytic, or algebraic, trace morphism $\mathrm{Tr}_f : K_X \to K_Y$. Then the compatibility with the differential of L is clear, if it is defined in $\mathcal{H}om^f_{\mathrm{Diff}}$. For the proof of the isomorphism we may assume that L is a coherent $\mathcal{O}_X$-Module. Then (0.5) is quasi-isomorphic to the composition :

$$f_\bullet \mathcal{H}om_{\mathcal{O}_X}(L, K_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathbb{R}f_\bullet L, f_\bullet K_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathbb{R}f_\bullet L, K_Y) \tag{0.6}$$

and the assertion is reduced to the dualities in [H], [RRV]. Here $\mathbb{R}f_\bullet L$ is defined by choosing some canonical $f_\bullet$-acyclic resolution of L.

We also show the compatibility of the dual functor with the de Rham functor $DR\mathbb{D} = \mathbb{D}DR$ in the holonomic case using the forgetful functor of the differential complexes, where the proof of the isomorphism is same as in [K1]. Then the compatibility of the duality for proper morphisms (0.1) with the topological (*i.e.* Verdier) duality by the de Rham functor becomes trivial, because $\mathrm{Tr}_f : \widetilde{K}_X \to \widetilde{K}_Y$ represents the topological trace morphism $\mathrm{Tr}_f : \mathbb{R}f_\bullet \mathbb{C}_X[2d_X] \to \mathbb{C}_Y[2d_Y]$.

These results were first proved in the filtered case in [S1, paragraph 2], where the arguments are sometimes simplified (*e.g.* the stability of the coherence by the direct image by proper morphisms) due to the existence of the global filtration. In fact, we have to use the filtrered theory to relate the two trace morphisms $\mathrm{Tr}_f : \widetilde{K}_X \to \widetilde{K}_Y$ and $\mathrm{Tr}_f : K_X \to K_Y$ (*see* 3.7).

Here it should be noted that

$$F_p \mathcal{H}om_{\mathrm{Diff}}(L, L') := \mathcal{H}om_{\mathcal{O}_X}(L, L' \otimes F_p\mathcal{D}_X)$$

the sheaf of the differential morphisms of order $\leq p$ coincides with $\mathcal{D}iff^p_X(L, L')$ the differential operators of order $\leq p$ in the sense of GROTHENDIECK [BO] (*see* (1.20.2)).

We also introduce the notion of diagonal pairing to simplify some argument in the proof of the fully faithfulness of the Riemann-Hilbert correspondence, *cf.* 4.7. This can be also used to define the duality isomorphism, *cf.* 4.8, and might simplify some arguments which should be needed in the proof of [Sc 1–2] (*see* 4.9).

In paragraph 1 we define the induced $\mathcal{D}$-Modules and the differential complexes, and prove some equivalence of categories to assure the existence of some resolution. In paragraph 2 we define the dual and prove the compatibility with the de Rham functor in the holonomic case. In paragraph 3 we show the duality for proper morphisms and its compatibility with the topological one. In paragraph 4 we explain about the diagonal pairings.

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1. Induced $\mathcal{D}$-Modules and Differential Complexes

1.0. — In this note X denotes a complex manifold, or a smooth algebraic variety, and $\mathcal{D}_X$ the sheaf of holomorphic, or algebraic, differential operators. In the algebraic case, the $\mathcal{O}_X$- (and D_X-) Modules are assumed quasi-coherent (except in paragraph 4). We identify the left and right $\mathcal{D}_X$-Modules by the functor $M \mapsto \Omega_X^{d_X} \otimes_{\mathcal{O}_X} M$, where $d_X := \dim X$. We use mainly the right $\mathcal{D}$-Modules, because they are more convenient to the definition of dual and the proof of the duality (in fact, the induced $\mathcal{D}$-Modules are naturally defined as right $\mathcal{D}$-Modules).

1.1 *Definition.* — A $\mathcal{D}_X$-Module M is *induced*, if it is isomorphic to $L \otimes_{\mathcal{O}_X} \mathcal{D}_X$ for an $\mathcal{O}_X$-Module L. $M_i(\mathcal{D}_X)$ denotes the additive category of the induced $\mathcal{D}_X$-Modules, which is a full sub category of the abelian category of $\mathcal{D}_X$-Modules $M(\mathcal{D}_X)$. Then $C_i^b(\mathcal{D}_X)$, $K_i^b(\mathcal{D}_X)$, $D_i^b(\mathcal{D}_X)$ and $C^b(\mathcal{D}_X)$, *etc.* (same for C_i, C_i^+, C_i^-, *etc.*) are defined as in [V1], where $D_i^b(\mathcal{D}_X)$, $D^b(\mathcal{D}_X)$, *etc.* are obtaind by inversing the quasi-isomorphisms in $K_i^b(\mathcal{D}_X)$, $K^b(\mathcal{D}_X)$, *etc.*

1.2 Lemma. — *For induced $\mathcal{D}_X$-Modules M, N, we have $M \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X = M \otimes_{\mathcal{D}_X} \mathcal{O}_X$, i.e. $\mathcal{T}or_i^{\mathcal{D}_X}(M, \mathcal{O}_X) = 0$ for $i \neq 0$, and the natural morphism*

$$\mathcal{H}om_{D_X}(M,N) \longrightarrow \mathcal{H}om_{\mathbb{C}_X}\big(M \otimes_{\mathcal{D}_X} \mathcal{O}_X, N \otimes_{\mathcal{D}_X} \mathcal{O}_X\big) \tag{1.2.1}$$

is injective.

Proof. — The first assertion is clear, and the second is shown in [S1, 2.2.2].

1.3 *Definition.* — For $\mathcal{O}_X$-Modules L, L', we denote by $\mathcal{H}om_{\mathrm{Diff}}(L, L')$