BULLETIN DE LA S. M. F.

KATHERINE NORRIE

Actions and automorphisms of crossed modules

Bulletin de la S. M. F., tome 118, nº 2 (1990), p. 129-146
<http://www.numdam.org/item?id=BSMF_1990__118_2_129_0>

Bull. Soc. math. France,
118, 1990, p. 129–146.

ACTIONS AND AUTOMORPHISMS OF CROSSED MODULES

BY

KATHERINE NORRIE (*)

RÉSUMÉ. — Il est bien connu que le groupe des automorphismes d'un groupe N fait partie d'un module croisé $N \to \operatorname{Aut} N$. Dans cet article, on étudie des structures plus compliquées, par exemple les *carrés croisés*, qui correspondent aux automorphismes des modules croisés. On peut ainsi étudier les actions et les produits semi-directs des modules croisés.

ABSTRACT. — It is standard that the automorphisms of a group N fit into a crossed module $N \to \operatorname{Aut} N$. This paper explores the corresponding more elaborate structures, for example *crossed squares*, into which fits the automorphism group of a crossed module. This also enables accounts of actions and semi-direct products of crossed modules.

1. Introduction

Crossed modules have been used widely, and in various contexts, since their definition by J.H.C. WHITEHEAD [24, 26] in his investigation of the algebraic structure of second relative homotopy groups. Areas in which crossed modules have been applied include the theory of group presentations (see the survey [3]), algebraic K-theory [12], and homological algebra [10, 18]. Now crossed modules can be viewed as 2-dimensionnal groups [1] and it is therefore of interest to consider counterparts for crossed modules of concepts from group theory; in this paper we shall generalize some aspects of the theory of automorphisms from groups to crossed modules. The surprise is the ease with which the theory transcribes, which in effect confirms the above view of crossed modules.

The automorphism group $\operatorname{Aut} N$ of a group N comes equipped with the canonical homomorphism $\tau : N \to \operatorname{Aut} N$ which has image $\operatorname{Inn} N$, the

(*) Texte reçu le 17 mars 1989, révisé le 6 mars 1990.
K. NORRIE, School of Mathematics, Univ. of Wales, Bangor, Gwynedd, LL57 1UT, Grande Bretagne and Dept. of Computer Sc., Royal Holloway and Bedford New College, Egham Hill, Egham, Surrey TW 20 0EX, Grande Bretagne.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037–9484/1990/129/$ 5.00

group of inner automorphisms of N, and kernel $Z(N)$, the centre of N. The quotient $\operatorname{Aut} N / \operatorname{Inn} N$ is the group of outer automorphisms of N, denoted by $\operatorname{Out} N$. We note that τ is one of the standard examples of a crossed module. Now if $1 \to N \to G \to Q \to 1$ is a short exact sequence of groups, then there is a homomorphism $\theta : G \to \operatorname{Aut} N$ making commutative the following diagram

$$(*) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow & & \\ 1 & \longrightarrow & \operatorname{Inn} N & \longrightarrow & \operatorname{Aut} N & \longrightarrow & \operatorname{Out} N & \longrightarrow & 1. \end{array}$$

However, in some other familiar categories, the set of structure preserving self-maps of a given object will not fulfil the role just delineated for the automorphism group $\operatorname{Aut} N$. In the category of groups, the automorphism group plays a dual role of capturing the notions of action and of structure preserving self-maps. In other categories these notions do not necessarily coincide, and for our present purposes it is the notion of action that is significant. We shall define *actor crossed modules*, and show how they provide an analogue of automorphism groups of groups. We establish that a version of (*) holds for actor crossed modules. We use this actor to define *actions* of crossed modules, and we establish the main properties of these. In particular, we construct semi-direct products of crossed modules, and the holomorph of a crossed module.

Part of the motivation for this work arose from analogies between groups and algebras. In the category of associative algebras, the appropriate replacement for the automorphism group is the bimultiplication algebra [15]; for Lie algebras we must employ the derivation algebra [14].

We shall also show that relationships between groups and crossed modules, involving the automorphism group, are mirrored in corresponding relationships between crossed modules and their three-dimensional analogues, crossed squares. Crossed squares were first defined by D. Guin-Walery and J.-L. Loday in [9], where they are applied to problems in algebraic K-theory. Some applications of crossed squares in homotopy theory may be found in [13, 4, 5]. Since the homomorphism $\tau : N \longrightarrow \operatorname{Aut} N$ is a crossed module, we might expect that a crossed module and its actor will give rise to a crossed square. We show that this is indeed the case. This allows us to relate actors of crossed modules with the equivalence between crossed squares and 2-cat-groups, due to Guin-Walery and Loday, as described in [13].

Acknowledgements

The results in this paper formed part of my Ph.D. thesis [19] written at King's College, London, and supported by a research studentship from the Science and Engineering Research Council. I would like to thank my supervisor, Dr. A.S.-T. Lue for his advice and assistance during my period of study. I am grateful to Professor R. Brown and the Pure Mathematics Department of the University College of North Wales, where this paper was written, for their hospitality in 1987. I would also like to thank Dr. N.D. Gilbert for help in preparing this version of this paper publication, and Prof. Brown and a referee for helpful comments.

1. The actor and centre of a crossed module

Recall that a *crossed module* (T, G, ∂) consist of a group homomorphism $\partial : T \to G$ called the *boundary map*, together with an action $(g, t) \mapsto {}^{g}t$ of G on T satisfying

1) $\partial({}^{g}t) = g\,\partial(t)\,g^{-1}$,
2) ${}^{\partial(s)}t = s\,t\,s^{-1}$,

for all $g \in G$ and $s, t \in T$. In addition to the inner automorphism map $\tau : N \to \operatorname{Aut} N$ already mentioned, other standard examples of crossed modules are :

- a G-module M with the zero homomorphism $M \to G$;
- the inclusion of a normal subgroup $N \to G$;
- and any epimorphism $E \to G$ with central kernel.

There are two canonical ways in which a group G may be regarded as a crossed module : via the identity map $G \to G$ or via the inclusion of the trivial subgroup.

We say that (S, H, ∂') is a *subcrossed module* of the crossed module (T, G, ∂) if

i) S is a subgroup of T, and H is a subgroup of G;
ii) ∂' is the restriction of ∂ to S, and
iii) the action of H on S is induced by the action of G on T.

A subcrossed module (S, H, ∂) of (T, G, ∂) is *normal* if

i) H is a normal subgroup of G;
ii) ${}^{g}s \in s$ for all $g \in G$, $s \in S$, and
iii) ${}^{h}t\,t^{-1} \in S$ for all $h \in H$, $t \in T$.

A crossed module *morphism* $\langle \alpha, \phi \rangle : (T, G, \partial) \longrightarrow (T', G', \partial')$ is a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & M \\ {\scriptstyle\partial}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ G & \xrightarrow{\phi} & P \end{array}$$

such that for all $x \in G$ and $t \in T$, we have $\alpha({}^{x}t) = {}^{\phi(x)}\alpha(t)$. We say that $\langle\alpha, \phi\rangle$ is an *isomorphism* if α and ϕ are both isomorphisms; similarly, we define *monomorphisms, epimorphisms* and *automorphisms* of crossed modules. We denote the group of automorphisms of (T, G, ∂) by $\mathrm{Aut}(T, G, \partial)$. The *kernel* of the crossed module morphism $\langle\alpha, \phi\rangle$ is the normal subcrossed module $(\ker\alpha, \ker\phi, \partial)$ of (T, G, ∂), denoted by $\ker\langle\alpha, \phi\rangle$. The *image* $\mathrm{im}\langle\alpha, \phi\rangle$ of $\langle\alpha, \phi\rangle$ is the subcrossed module $(\mathrm{im}\,\alpha, \mathrm{im}\,\phi, \mu')$ of (M, P, μ).

The *trivial* crossed module (1,1,1) will be written simply as 1. We shall occasionally supress explicit mention of the boundary map in a crossed module (T, G, ∂) and write simply (T, G).

For a crossed module (T, G, ∂), denote by $\mathrm{Der}(G, T)$ the set of all derivations from G to T, i.e. all maps $\chi : G \to T$ such that for all $x, y \in G$,

$$\chi(xy) = \chi(x)\, {}^{x}\chi(y).$$

Each such derivation χ defines endomorphisms σ $(= \sigma_\chi)$ and θ $(= \theta_\chi)$ of G, T respectively, given by

$$\sigma(x) = \partial\chi(x)x \qquad \theta(t) = \chi\partial(t)t$$

Clearly,

$$\sigma\partial(t) = \partial\theta(t), \quad \theta\chi(x) = \chi\sigma(x), \quad \theta({}^{x}t) = {}^{\sigma(x)}\theta(t).$$

Following WHITEHEAD [25], we define a multiplication in $\mathrm{Der}(G, T)$ by the formula $\chi_1 \circ \chi_2 = \chi$, where

$$\chi(x) = \chi_1\sigma_2(x)\chi_2(x) \quad \big(= \theta_1\chi_2(x)\chi_1(x)\big).$$

This turns $\mathrm{Der}(G, T)$ into a semigroup, with identity element the derivation which maps each element of G into the identity element of T. Moreover, if $\chi = \chi_1 \circ \chi_2$ then $\sigma = \sigma_1\sigma_2$. The *Whitehead group* $D(G, T)$ is defined to be the group of units of $\mathrm{Der}(G, T)$, and the elements of $D(G, T)$ are called *regular derivations.*

The following proposition combines results from [17] and [25].