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SUBELLIPTIC VARIATIONAL PROBLEMS

BY

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RÉSUMÉ. — En utilisant la méthode directe et l'itération de MOSER, nous démontrons l'existence et la C^μ-régularité du point stationnaire pour le problème variationnel elliptique dégénéré $I(\mu) = \int_\Omega F(x,u,Xu)\,dx$ où $X = (X_1,\dots,X_m)$ est un système de champs de vecteurs C^∞ réels qui satisfait à la condition de Hörmander. Les hypothèses sur $F(x,u,\xi)$ sont analogues à celles faites pour les problèmes elliptiques.

ABSTRACT. — Using the direct method and the MOSER's process, we prove the existence and C^μ regularity of stationary point for the degenerate elliptic variational problem $I(\mu) = \int_\Omega F(x,u,Xu)\,dx$ where $X = (X_1,\dots,X_m)$ is a system of real smooth vector fields which satisfy the Hörmander's condition. The assumption imposed on $F(x,u,\xi)$ are similar to those for the elliptic case.

1. Introduction

In this paper, we study the existence and the regularity for the minimum points of the following variational problem :

$$I(\mu) = \int_\Omega F(x,u,Xu)\,dx, \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, and $X = (X_1,\dots,X_m)$ is a system of real smooth vector fiels in M, which is a bounded domain of $\mathbb{R}^n$ such that $\Omega \subset\subset M$. We assume that $F(x,u,\xi)$ is convex in ξ and that X satisfy the Hörmander's condition in M, i.e.

$$\text{(H)} \qquad \begin{cases} \{X_j\} \textit{ together with their commutators} \\ \textit{up to a certain fixed length } r \textit{ span the} \\ \textit{tangent space at each point of } M. \end{cases}$$

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In this case, the Euler's equation of (1.1)

$$\sum_{j=1}^{m} X_j^* F_{\xi_j}(x,u,Xu) + F_u(x,u,Xu) = 0 \tag{1.2}$$

is degenerately elliptic. We assume also, for $j = 1, \dots, m$,

$$\mathrm{Mes}\{x \in \Omega \mid X_j(x) = 0\} = 0.$$

For linear problems of this kind, there is a lot of work after the first appearing of L. HÖRMANDER's (see [1, 2, 4, 5, 7, 8, 9]). In particular, we note that the Hörmander's condition permit us to define a metric $\rho(x,y)$ associated with X in M. Using the geometry of this metric, we can think the Hörmander operator

$$H = \sum_{j=1}^{m} X_j^2 + c(x)$$

as the Laplace operators. Then we can study the existence of weak stationary points of (1.1) by the direct method just as we do for the elliptic problem, and discuss the C^μ regularity of weak solution of (1.2) by Moser's process just as we do for the linear degenerate elliptic problems.

Our result is an extention of those for the elliptic variationnal problem to a certain class of highly degenerate problems. We will consider the C^∞ regularity problems in another paper.

2. Function space $\mathbf{M^{k,p}(\Omega)}$

In order to study the weak solution, we introduce a function space $M^{k,p}(\Omega)$ associated with X, which is analogue to Sobolev's space. For any integer $k \geq 1$, $p \geq 1$ and $\Omega \subset\subset M$, we define

$$M^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \mid X^J f \in L^p(\Omega),\ \forall J = (j_1, \dots, j_s),\ |J| \leq k \Big\} \tag{2.1}$$

where $X^J f = X_{j_1} \dots X_{j_s} f$, $|J| = s$ and define the norm in $M^{k,p}(\Omega)$ to be

$$||f||_{M^{k,p}(\Omega)} = \Big(\sum_{|J| \leq k} ||X^J f||^p_{L^p(\Omega)} \Big)^{1/p}. \tag{2.2}$$

We also denote by $M^k(\Omega) = M^{k,2}(\Omega)$. Then we have :

THEOREM 1. — *The function space $M^{k,p}(\Omega)$ is a Banach space for $1 \le p < +\infty$, which is reflexive for $1 < p < +\infty$ and separable for $1 \le p < +\infty$. Also, $M^k(\Omega)$ is a separable Hilbert space.*

Proof. — a) Let $J = (j_1, \ldots, j_s)$, with $1 \le j_c \le m$, and denote by X^{J*} the adjoint operator of X^J. Then

$$M^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \mid \exists g_J \in L^p(\Omega) \text{ such that } \int_\Omega f \cdot X^{J*}\varphi\, dx = \int_\Omega g_J\, \varphi dx,\ \ \varphi \in C_0^\infty(\Omega),\ \ |J| \le k \Big\}. \tag{2.3}$$

Suppose $\{u_j\}$ to be a Cauchy sequence of $M^{k,p}(\Omega)$, then $\{X^J u_j\}$, for $|J| \le k$, are all Cauchy sequence in $L^p(\Omega)$. Hence there exists $u^J \in L^p(\Omega)$ such that $X^J u_j \to u^J$ in $L^p(\Omega)$. On the other hand

$$\int_\Omega u_j X^{J*}\varphi\, dx = \int_\Omega X^J u_j \varphi\, dx, \quad \varphi \in C_0^\infty, \ |J| \le k.$$

Let $j \to \infty$, we have

$$\int_\Omega u^0 X^{J*}\varphi\, dx = \int_\Omega u^J \varphi\, dx, \quad \varphi \in C_0^\infty(\Omega), \ |J| \le k,$$

which proves $u^0 \in M^{k,p}(\Omega)$, $X^J u^0 = u^J$ and $||u_j - u^0||_{M^{k,p}(\Omega)} \to 0$.

b) Setting $E = \prod_{|J|\le k} L^p(\Omega)$, then E is a reflexive Banach space for $1 < p < +\infty$. Define $T : M^{k,p}(\Omega) \to E$ by $Tu = (X^J u)$, then T is an isometry from $M^{k,p}(\Omega)$ to E. Since $T(M^{k,p}(\Omega))$ is a closed subspace of E and $T(M^{k,p}(\Omega))$ is reflexive, then $M^{k,p}(\Omega)$ is also reflexive. The proof for separability is similar.

We denote by $M_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $M^{k,p}(\Omega)$. From the subellipticity of Hörmander's operator H, we have the following lemma :

LEMMA 2. — *Let Ω be a bounded subdomain of M. Assume that X satisfies the Hörmander's condition in M. Then, we have the continuous imbedding $M_0^{k,p}(\Omega) \subset W^{k/r,p}(\Omega)$ for all $k \ge 1$, $p \ge 1$ and there exists $C = C(p, \Omega, r)$ such that*

$$||u||_{W^{k/r,p}}(\Omega) \le C ||u||_{M^{k,p}}(\Omega) \tag{2.4}$$

for all $u \in M_0^{k,p}(\Omega)$. (Here, $W^{s,p}(\Omega)$ is the usual Sobolev's space.)

For the proof of this LEMMA, see [2, 9]. Using the classical Sobolev inequality in $W^{s,p}(\Omega)$ and imbedding LEMMA above, we obtain the following Sobolev inequality for the function space $M^{k,p}(\Omega)$.

THEOREM 3. — *Assume that Ω is a C^∞ domain. Then, we have continuous imbedding*

$$M_0^{k,p}(\Omega) \subset \begin{cases} L^{np/(n-kp/r)}(\Omega) & \text{for } kp < nr, \\ C^m(\overline{\Omega}) & \text{for } k/r - n/p > m \geq 0. \end{cases} \tag{2.5}$$

Further, there exists a constant $C = C(n, r, p, k)$ such that for any $u \in M_0^{k,p}(\Omega)$ we have

$$\begin{cases} ||u||_{L^{np/(n-kp/r)}(\Omega)} \leq C||u||_{M^{k,p}(\Omega)} & \text{for } kp < nr, \\ ||u||_{C^m(\overline{\Omega})} \leq C|\Omega|^{k/n-r/p}||u||_{M^{k,p}(\Omega)} & \text{for } k/r - n/p > m \geq 0. \end{cases} \tag{2.6}$$

By a contradiction argument based on the compactness result of the usual Sobolev's space, we obtain an interpolation inequality for the space $M^{k,p}(\Omega)$.

LEMMA 4. — *Assume that Ω is a C^∞ subdomain of M and u an element of $M^{k,p}(\Omega)$. Then, for any $\varepsilon > 0$ and $0 < |J| < k$, we have*

$$||X^J u||_{L^p(\Omega)} \leq \varepsilon||u||_{M^{k,p}(\Omega)} + C||u||_{L^p(\Omega)}$$

where $C = C(k, \Omega, \varepsilon)$.

We define now a metric $\rho(x, y)$ associated with X in M as in [7, 9], and take

$$B_R(x) = \{y \in \Omega \mid \rho(x, y) < R\}$$

for $R > 0$ small enough. Then, in the function space $M^{k,p}(\Omega)$, we have also the following Poincaré inequality.

LEMMA 5

(1) *For any $x^0 \in \Omega$, there exists $R_0 > 0$ such that for all $0 < R \leq R_0$, if $\varphi \in M_0^{1,p}(B_R(x^0))$, then*

$$||\varphi||_{L^p(B_R(x^0))} \leq CR||X\varphi||_{L^p(B_R(x^0))} \tag{2.8}$$

where C is of independant on φ and R.

(2) *If, in the system of vector field $X = (X_1, \ldots, X_m)$ there exists at last one vector field which can be globally straightened in Ω, then we have*

$$||\varphi||_{L^p(\Omega)} \leq C \operatorname{diam} \Omega||X\varphi||_{L^p(\Omega)} \tag{2.9}$$

for all $\varphi \in M_0^{1,p}(\Omega)$, where C is of independant on φ and Ω.