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RINGS OF DIFFERENTIAL OPERATORS OVER RATIONAL AFFINE CURVES

BY

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RÉSUMÉ. — Soit X une courbe algébrique irréductible sur $\mathbb{C}$ dont la normalisée est la droite affine et telle sur le morphisme de normalisation est injectif. Soit $D(X)$ l'anneau des opérateurs différentiels sur X. Nous étudions un invariant pour l'anneau $D(X)$ des opérateurs différentiels sur X, noté $\operatorname{codim} D(X)$. En particulier, nous montrons que $D(X) \cong D(Y)$ implique $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$. Cela permet de distinguer dans certains cas les anneaux d'opérateurs différentiels de courbes non-isomorphes. En outre, nous décrivons les sous-algèbres ad-nilpotentes maximales de $D(X)$. Nous montrons que si B est une sous-algèbre ad-nilpotente maximales de $D(X)$, alors B est un sous-anneau de type fini d'un $\mathbb{C}[b]$ où b désigne un élément du corps des fractions de $D(X)$; de plus, la clôture intégrale de B est $\mathbb{C}[b]$.

ABSTRACT. — Let X be an irreducible algebraic curve over the complex numbers such that its normalization is the affine line, and the normalization map is injective. Let $D(X)$ denote its ring of differential operators. We find an invariant for $D(X)$ denoted as $\operatorname{codim} D(X)$. In particular, we show that $D(X) \cong D(Y)$ implies $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$. This allows us to distinguish certain rings of differential operators of non-isomorphic curves. We also describe the maximal ad-nilpotent subalgebras of $D(X)$. We show that if B is a maximal ad-nilpotent subalgebra of $D(X)$, then B is a finitely generated subring of $\mathbb{C}[b]$ for some element b of the quotient field of $D(X)$ and the integral closure of B is $\mathbb{C}[b]$.

1. Introduction

Let X and Y be irreducible algebraic curves over the complex numbers, $\mathbb{C}$. Let $D(X)$ and $D(Y)$ denote their ring of differential operators, respectively. (For definition see [9]). This paper is motivated by the following open question.† Does $D(X) \cong D(Y)$ imply that $X \cong Y$? Let $\widetilde{X}$ denote

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† G. LETZTER has now found nonisomorphic curves X and Y with isomorphic rings of differential operators (see [4]).

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the normalization of X. MAKAR–LIMANOV [5] shows that the set of ad-nilpotent elements $N(X)$ is exactly $O(X)$ whenever $O(X)$ is not a subring of a polynomial ring in one variable over $\mathbb{C}$. He thus answers the question affirmatively for these curves. Let $\mathbb{A}^1$ denote the affine line. PERKINS [8] extends this result showing that $D(X) \cong D(Y)$ implies $X \cong Y$ whenever $\widetilde{X} \neq \mathbb{A}^1$, or $\widetilde{X} = \mathbb{A}^1$ but the normalization map $\pi : \widetilde{X} \to X$ is not injective. Thus, in the paper, we are interested in curves X such that $\widetilde{X} \cong \mathbb{A}^1$ and $\pi : \widetilde{X} \to X$ is injective. STAFFORD [10] shows the conjecture holds the following two examples of such curves : when X is the affine line $\mathbb{A}^1$, or when X is the cubic cusp $y^2 = x^3$.

For the remainder of the paper, assume that X is a curve such that its normalization is isomorphic to the affine line $\mathbb{A}^1$ with an injective normalization map. We may therefore assume that the coordinate ring of X, denoted $O(X)$, is a subring of a polynomial ring in one variable $\mathbb{C}[x]$ such that the integral closure of $O(X)$, written $\widetilde{O(X)}$, is equal to $C[x]$. Furthermore $D(X)$ is a subring of $\mathbb{C}(x)[\partial]$ where $[\partial, x] = 1$. Here ∂ is just $\partial/\partial x$ and the element $f_n(x)\partial^n + \cdots + f_0(x)$ of $D(X)$ sends $g(x) \in O(X)$ to $f_n(x)g^{(n)}(x) + \cdots + f_0(x)g(x)$ where $g^{(n)}(x)$ denotes the n^{th} derivative of $g(x)$.

PERKINS studies rings that satisfy these conditions in [8]. He shows that in many cases, $D(X)$ contains maximal commutative ad-nilpotent subalgebras not isomorphic to $O(X)$. Thus, for these curves, the set $N(X)$ of ad-nilpotent elements does not determine $O(X)$.

In this paper, we obtain an invariant for $D(X)$ and a nice description of the maximal ad-nilpotent subalgebras of $D(X)$. Set $T = \mathbb{C}(x)[\partial]$ and set $\partial\text{-deg}\, w = n$ where $w = f_n(x)\partial^n + \cdots + f_0(x)$ is an element of T. Define a filtration on T by $T_i = \{w \in T \mid \partial\text{-deg}\, w \leq i\}$ and hence on any subring R of T by $R_i = R \cap T_i$. (Note that this is the same filtration on $D(X)$ as the one defined by the order of the differential operator.) We may form the associated graded ring $\partial\text{-gr}\, R = \bigoplus R_i/R_{i-1}$. We define $\operatorname{codim} R$ to be equal to $\dim_{\mathbb{C}} \partial\text{-gr}\, \mathbb{C}[x, \partial]/\partial\text{-gr}\, R$ for those subrings R of T such that $\partial\text{-gr}\, R \subset \partial\text{-gr}\, \mathbb{C}[x, \partial]$.

Now assume that both X and Y are affine curves with normalization equal to the affine line and injective normalization map. By [9], both $\partial\text{-gr}\, D(X)$ and $\partial\text{-gr}\, D(Y)$ are subrings of $\partial\text{-gr}\, \mathbb{C}[x, \partial]$ and $\operatorname{codim} D(X)$ and $\operatorname{codim} D(Y)$ are finite numbers.

Our main results are :

THEOREM. — *Suppose that B is a maximal* ad-*nilpotent subalgebra of $D(X)$. Then there exists elements x' and ∂' in the quotient field of*

$\mathbb{C}(x)[\partial]$ such that $[\partial', x'] = 1$, $D(X)$ is a subring of $\mathbb{C}(x')[\partial']$, $D(X) \cap \mathbb{C}(x') = B$, and the integral closure of B is $\mathbb{C}[x']$. Furthermore, ∂'-$\operatorname{gr} D(X)$ is a subring of ∂'-$\operatorname{gr} \mathbb{C}[x', \partial']$ and

$$\dim_{\mathbb{C}} \partial'\text{-}\operatorname{gr} \mathbb{C}[x', \partial'] / \partial'\text{-}\operatorname{gr} D(X) = \operatorname{codim} D(X).$$

COROLLARY. — *If $D(X) \cong D(Y)$, then* $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$.

This result permits one to distinguish many rings of differential operators. For example, set $O(X_n) = \mathbb{C} + x^n\mathbb{C}[x]$. Then it will follow from the COROLLARY, that $D(X_n) \cong D(X_m)$ implies that $n = m$.

2. Graded Algebras of $D(X)$

In this section, α and β are nonnegative real numbers with $\alpha + \beta > 0$. Define valuations $V_{\alpha,\beta}$ on $\mathbb{C}(x)[\partial]$ as follows. Set

$$V_{\alpha,\beta}\Big(w_n(x)\partial^n + w_{n-1}(x)\partial^{n-1} + \cdots + w_0(x)\Big)$$

equal to $\max\{\alpha d_m + \beta m \mid n \geq m \geq 0\}$ where $d_m = \deg(w_n(x))$. This extends the notion of valuations introduced by DIXMIER in [2] for the Weyl algebra. For each valuation $V_{\alpha,\beta}$ we may define a filtration of $\mathbb{C}(x)[\partial]$, and hence on any subring R of $\mathbb{C}(x)[\partial]$ as follows. Recall that $T = \mathbb{C}(x)[\partial]$. Set $T_i = \{z \in T \mid V_{\alpha,\beta}(z) \leq i\}$ and $R_i = R \cap T_i$. We may then define the associated graded algebra $\operatorname{gr}_{\alpha,\beta} R = \bigoplus R_i/R_{i-1}$. Now the commutator $[x^i\partial^j, x^k\partial^\ell] = (kj - i\ell)x^{i+k-1}\partial^{j+\ell-1}+$ terms with x-degree less than $i + k - 1$ and ∂-degree less than $j + \ell - 1$. Therefore $V_{\alpha,\beta}([x^i\partial^j, x^k\partial^\ell]) < \alpha(i + k) + \beta(\ell + j)$. It follows that $\operatorname{gr}_{\alpha,\beta}(\mathbb{C}(x)[\partial])$ is a commutative algebra.

Note that when $\alpha = 0$ and β is positive, then the filtration defined by $V_{0,\beta}$ on $D(X)$ is the same filtration on $D(X)$ as the one defined by ∂-deg in the introduction. We will write ∂-$\operatorname{gr} D(X)$ for $\operatorname{gr}_{0,\beta} D(X)$ and ∂-deg for $V_{0,\beta}$. Similarly, when $\beta = 0$ and α is positive the graded algebra determined by $V_{\alpha,0}$ is the same as x-$\operatorname{gr} R$ determined by x-deg defined in [8].

Set $\operatorname{gr}_{\alpha,\beta} x = x$ and $\operatorname{gr}_{\alpha,\beta} \partial = y$. Since $D(\widetilde{X})$ is just the first Weyl algebra, A_1, we have that ∂-$\operatorname{gr} D(\widetilde{X}) = \mathbb{C}[x, y]$ where ∂-$\operatorname{gr} x = x$ and ∂-$\operatorname{gr} \partial = y$. By [9, Proposition 3.11], it follows that ∂-$\operatorname{gr} D(X)$ is a subring of $\mathbb{C}[x, y]$ and by [8, Lemma 2.3], x-$\operatorname{gr} D(X)$ is also a subring of $\mathbb{C}[x, y]$. In the following lemma, we extend this to other gradings.

LEMMA 2.1. — *Let R be a subring of $\mathbb{C}(x)[\partial]$ such that ∂-$\operatorname{gr} R \subset \mathbb{C}[x, y]$. Then the graded algebra $\operatorname{gr}_{\alpha,\beta} R$ is a subring of $\mathbb{C}[x, y]$.*

Proof. — If $\alpha = 0$ then $\mathrm{gr}_{\alpha,\beta} R = \partial\text{-gr}\, R$. So we may assume that α is positive. Let w be a typical element of $D(X)$. Write $w = g_m(x)\partial^m + \cdots + g_0(x)$ where $g_i(x) \in \mathbb{C}(x)$ for $0 \leq i \leq m$. Set degree of $g_i(x)$ equal to d_i for $0 \leq i \leq m$. Since $\partial\text{-gr}\, R \subset \mathbb{C}[x,y]$, it follows that $g_m(x) \subset \mathbb{C}[x]$ and thus $d_m \geq 0$. Set $N = V_{\alpha,\beta}(w)$. By the definition of $V_{\alpha,\beta}$, it follows that $N = \max\{d_i\alpha + i\beta \mid 0 \leq i \leq m\}$. Hence $\mathrm{gr}_{\alpha,\beta}(w) = \sum_{0\leq s\leq m} \gamma_s x^{d_s} y^s$ where $\gamma_s = 0$ if $V_{\alpha,\beta}(x^{d_s}\partial^s) < N$, and $\gamma_s x^{d_s}$ is the leading term of $g_s(x)$ if $V_{\alpha,\beta}(x^{d_s}\partial^s) = N$. We need to show that whenever $\gamma_s \neq 0$, we have $x^{d_s}y^s \in \mathbb{C}[x,y]$. In particular, since $0 \leq s \leq m$, we need to show that $d_s \geq 0$ whenever $\gamma_s \neq 0$. Now $N = V_{\alpha,\beta}(w) \geq V_{\alpha,\beta}(g_m(x)\partial^m) = d_m\alpha + m\beta$. Hence $d_s\alpha + s\beta \geq d_m\alpha + m\beta$. Recall that $m \geq s, d_m \geq 0$, and that α is positive. It follows that $d_s \geq d_m \geq 0$. The lemma now follows.

Define a linear map $\phi : \mathbb{C}(x)[\partial] \to \mathbb{C}[x,\partial]$ as follows. Suppose that $w = g_m(x)\partial^m + \cdots + g_0(x)$ is an element of $\mathbb{C}(x)[\partial]$. For each i such that $1 \leq i \leq m$, there exists a unique polynomial $f_i(x)$ such that $\deg(g_i(x) - f_i(x)) < 0$. Set

$$\phi(w) = f_m(x)\partial^m + \cdots + f_0(x).$$

Now consider two rational functions $g_1(x)$ and $g_2(x)$ such that $\phi(g_1(x)) = f_1(x)$ and $\phi(g_2(x)) = f_2(x)$. Then clearly

$$\deg\Big(\lambda_1 g_1(x) + \lambda_2 g_2(x) - (\lambda_1 f_1(x) + \lambda_2 f_2(x)\Big) < 0 \quad \text{and}$$
$$\phi(\lambda_1 g_1(x) + \lambda_2 g_2(x)) = \lambda_1 f_1(x) + \lambda_2 f_2(x).$$

It follows that ϕ is a well defined linear map from $\mathbb{C}(x)[\partial]$ to $\mathbb{C}[x,\partial]$.

COROLLARY 2.2. — *Let R be a subring of $\mathbb{C}(x)[\partial]$ such that $\partial\text{-gr}\, R \subset \mathbb{C}[x,y]$. If w is an element of R, then $\mathrm{gr}_{\alpha,\beta}\, \phi(w) = \mathrm{gr}_{\alpha,\beta}(w)$.*

Proof. — This is clear since $\mathrm{gr}_{\alpha,\beta}(w - \phi(w))$ does not contain any monomials $x^{d_s}y^s$ with $d_s \geq 0$.

Remark 2.3. — Note that $\phi(R)$ is a linear subspace of the first Weyl algebra $A_1 = \mathbb{C}[x,\partial]$, but, generally speaking, is not a subalgebra. Nevertheless α,β gradings are defined on $\phi(R)$ and $\mathrm{gr}_{\alpha,\beta}\, \phi(R) = \mathrm{gr}_{\alpha,\beta}\, R$. Now

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, D(X) < \infty \quad \text{([9, 3.12]) and}$$
$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/x\text{-gr}\, D(X) < \infty \quad \text{([8, Lemma 2.5])}$$

In the next proposition, we will show that these two finite numbers are equal. We will later show that this codimension is an invariant for $D(X)$.