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JEAN-MICHEL BISMUT

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ETA INVARIANTS AND COMPLEX IMMERSIONS

BY

JEAN-MICHEL BISMUT (*)

RÉSUMÉ. — L'objet de cet article est de calculer l'invariant êta d'un complexe qui est acyclique en dehors d'une sous-variété. On utilise pour cela le théorème d'indice d'Atiyah-Patodi-Singer et les superconnexions de Quillen.

ABSTRACT. — The purpose of this paper is to calculate the eta invariant of a chain complex of vector bundles which is acyclic off a submanifold. The main tools are the index theorem of Atiyah-Patodi-Singer and the superconnections of Quillen.

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Let $i : M' \to M$ be an embedding of complex manifolds. Let μ be a holomorphic vector bundle on M', let

$$(\xi, v): \ 0 \longrightarrow \xi_m \xrightarrow{v} \xi_{m-1} \xrightarrow{v} \cdots \xi_0 \longrightarrow 0$$

be a holomorphic chain complex of vector bundles on M, which provides a resolution of the sheaf $i_*\mathcal{O}_{M'}(\mu)$. In particular the complex (ξ, v) is acyclic on $M \setminus M'$.

Let Z be an odd dimensional real compact spin submanifold of M which intersects M' so that $Z' = Z \cap M'$ is a submanifold of Z. Then the complex $(\xi, v)_{|Z}$ is acyclic off Z'.

Assume that $\xi_0, \ldots, \xi_m$ are equipped with Hermitian metrics and that $T_R Z$ is equipped with a scalar product. Then for $0 \leq k \leq m$, we can construct the reduced eta function $\overline{\eta}^{\xi_k}(s)$ of ATIYAH-PATODI-SINGER [APS] associated with the Dirac operator D^k acting on the spinors of $T_R Z$ twisted by $\xi_{k|Z}$. Set

$$\overline{\eta}^{\xi}(s) = \sum_0^m (-1)^k \overline{\eta}^{\xi_k}(s).$$

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J.-M. BISMUT , Université Paris-Sud, Département de Mathématique, Bât. 425, 91405 Orsay Cedex, France

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The purpose of this paper is to calculate (modulo integers) the linear combination of eta invariants $\overline{\eta}^{\xi}(0)$ in terms of a local object on Z and of an eta invariant on the submanifold Z'. In the case where Z is the boundary in M of a real submanifold Y, we give an explicit formula in THEOREM 2.9 which involves :

• A Chern-Simons current γ^Z on Z which is constructed using the results of BISMUT [B] and BISMUT-GILLET-SOULÉ [BGS1].

• A Chern-Simons form on Z' naturally associated with certain exact sequences involving the normal bundles to Z' in Z and in Y.

• An eta invariant on Z' associated with the Dirac operator on Z' acting on twisted spinors (where the twisting bundle involves the two normal bundles to Z' in Z and to M' in M explicitly).

We now make several comments on our formula.

• Our main result in THEOREM 2.9 still holds in ordinary real geometry. We work here in a complex geometric setting to be able to directly apply the results of [B]. The results of [B] have an obvious C^∞ analogue. The fact that (ξ, v) is a resolution of $\mathcal{O}_{M'}(\eta)$ implies that, by the local uniqueness of resolutions [E], [S], the complex (ξ, v) degenerates like a Koszul complex near M'. This would have to be introduced as a supplementary assumption in a C^∞ context.

• A more serious limitation of our result is that we assume the manifold Z bounds in M. However such results should hold in full generality, at the expense of more involved techniques. These will be developed elsewhere. The main interest of this paper is to give an explicit answer in a relatively simple case.

• Finally, let us point that our generalized Chern-Simons currents are directly related with the differential characters of CHEEGER and SIMONS [CSi].

In a second part of the paper, we consider a holomorphic submersion $\pi : M \to B$ which restricts to a submersion $\pi' : M' \to B$. Let $s \in S_1 \mapsto c_s \in B$ be a smooth loop. We then compare the holonomies of the direct image determinant line bundles $\lambda(\xi) = (\det R\pi_*(\xi))^{-1}$ and $\lambda(\mu) = (\det R\pi'_*(\mu))^{-1}$, when these line bundles are equipped with the holomorphic Hermitian connections associated with certain Quillen metrics [Q2], [BGS3]. Again, and for simplicity, we assume that the loop c bounds in B. Our result is then a straightforward application of the curvature calculations of BISMUT-GILLET-SOULÉ [BGS3]. The fact that such a result still holds even when c does not bound in B is now a consequence of a difficult result of BISMUT-LEBEAU [BL], whose proof is much more complicate than the one given here.

Also note that by a result of BISMUT-FREED [BF, Theorem 3.16], we know that the holonomy of certain connections on C^∞ determinant bundles of direct images can be evaluated in terms of adiabatic limits of eta invariants. Therefore our two main results, concerning eta invariants and determinants of direct images are intimately related, together with their possible extensions to the non bounding case. Also note that the relation of the holonomy theorem of [BF] to the differential characters of [CSi] has been developed in the context of direct images by GILLET and SOULÉ [GS].

This paper is organized as follows. In Section 1, we introduce our main assumptions and notations. In Section 2, we give a formula for $\overline{\eta}^\xi(0)$ in terms of the eta invariant of a submanifold and of a local quantity. Finally in Section 3, we give a relation which connects the holonomies of various determinant bundles over closed loops which bound.

The author is indebted to J. CHEEGER for helpful discussions concerning differential characters.

1. Complex immersions and resolutions

In this section we introduce our main assumptions and notations.

In a), we consider an immersion $i : M' \to M$ of complex manifolds, a holomorphic vector bundle μ on M' and a complex of vector bundles (ξ, v) which resolves $i_*\mu$ on M.

In b), we introduce various metrics on the considered vector bundles.

In c), we briefly describe the superconnection formalism of QUILLEN [Q1].

a) Complex manifolds and resolutions. — Let M be a compact connected complex manifold of complex dimension m. Let $M' = \bigcup_1^d M'_j$ be a finite union of disjoint compact connected complex submanifolds of M. Let i be the embedding $M' \to M$. Let N be the complex normal bundle to M' in M. Let

$$(1.1) \qquad (\xi, v) : 0 \longrightarrow \xi_m \xrightarrow{v} \xi_{m-1} \xrightarrow{v} \cdots \xi_0 \longrightarrow 0$$

be a holomorphic chain complex of vector bundles on the manifold M. Let μ be a holomorphic vector bundle on the manifold M'. We suppose there exists a holomorphic restriction map $r : \xi_{0|M'} \to \mu$.

We make the fundamental assumption that the sequence of sheaves

$$(1.2) \qquad 0 \longrightarrow \mathcal{O}_M(\xi_m) \xrightarrow{v} \cdots \xrightarrow{v} \mathcal{O}_M(\xi_0) \xrightarrow{r} i_*\mathcal{O}_{M'}(\mu) \longrightarrow 0$$

is exact. In particular, the complex (ξ, v) is acyclic on $M \setminus M'$.

For $x \in M'$, $0 \le k \le m$, let $H_{k,x}$ be the k^{th} homology group of the complex $(\xi, v)_x$. Set $H_x = \bigoplus_0^m H_{k,x}$.

The following results are consequences of the local uniqueness of resolution [Se, Chapter IV, Appendix 1], [E, Theorem 8] and are proved in [B, Section 1].

- For $k = 0, \dots, m$ the dimension of $F_{k,x}$ is constant on each M'_j, so that H_k is a holomorphic vector bundle on M'.
- For $x \in M'$, $U \in T_x M$, let $\partial_U v(x)$ be the derivative of the chain map v calculated in any given local holomorphic trivialization of (ξ, v) near x. Then $\partial_U v(x)$ acts on H_x. When acting on H_x, $\partial_U v(x)$ only depends on the image y of U in N_x. So we now write $\partial_y v(x)$ instead of $\partial_U v(x)$.
- For $x \in M'$, $y \in N$, $(\partial_y v)^2(x) = 0$. If $y \in N$, let i_y be the interior multiplication operator acting on the exterior algebra $\Lambda(N^*)$. The graded holomorphic complex $(H, \partial_y v)$ on the total space of the vector bundle N is canonically isomorphic to the Koszul complex $(\Lambda N^* \otimes \mu, i_y)$.

b) Assumption (A) on the Hermitian metrics of a chain complex. — We assume that $\xi_0, \dots, \xi_m$ are equipped with smooth Hermitian metrics $h^{\xi_0}, \dots, h^{\xi_m}$. We equip $\xi = \bigoplus_0^m \xi_k$ with the metric h^ξ which is the orthogonal sum of the metrics $h^{\xi_0}, \dots, h^{\xi_m}$. Let v^* be the adjoint of v with respect ot the metric h^ξ. By finite dimensional Hodge theory, we get an identification of smooth vector bundles on M'

$$H_k \cong \{f \in \xi_k \,;\ vf = 0 \,;\ v^* f = 0\}, \qquad 0 \le k \le m. \tag{1.3}$$

As a smooth subvector bundle of ξ_k, the right-hand side of (1.3) inherits a Hermitian metric from the metric h^{ξ_k} on ξ_k. Therefore H_k is a holomorphic Hermitian vector bundle on M'. Let h^{H_k} denote the Hermitian metric on H_k. We equip $H = \bigoplus_0^m H_k$ with the metric h^H which is the orthogonal sum of the metrics $h^{H_0}, \dots, h^{H_m}$.

Let g^N, g^μ be Hermitian metrics on N, η. We equip the vector bundle $\Lambda N^* \otimes \mu$ with the tensor product of the metric induced by g^N on ΛN^* and of the metric g^μ.

Definition 1.1. — Given metrics g^N, g^η on N, η, we will say that the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ verify assumption (A) with respect to g^N, g^μ if the canonical identification of holomorphic chain complexes on the total space of N

$$(H, \partial_y v) \cong (\Lambda N^* \otimes \mu, i_y) \tag{1.4}$$

also identifies the metrics.