BULLETIN DE LA S. M. F.

BARBARA FANTECHI

On the superadditivity of secant defects

Bulletin de la S. M. F., tome 118, n° 1 (1990), p. 85-100
<http://www.numdam.org/item?id=BSMF_1990__118_1_85_0>

Bull. Soc. math. France,
118, 1990, p. 85–100.

ON THE SUPERADDITIVITY OF SECANT DEFECTS

BY

BARBARA FANTECHI (*)

RÉSUMÉ. — Dans son article [Z2], M. ZAK a énoncé un théorème de super-additivité pour les défauts sécants de variétés projectives lisses ; ensuite, M. ÅDLANDSVIK a donné un contre-example. Dans cet article nous prouvons un théorème semblable à celui de ZAK, mais avec des hypothèses plus fortes ; nous remarquons que tous les corollaires énoncés par ZAK restent vrais, et que le contre-exemple de ÅDLANDSVIK ne satisfait pas les hypothèses supplémentaires.

ABSTRACT. — In his paper [Z2], ZAK stated a theorem of superadditivity for secant defects of smooth projective varieties; subsequently, ÅDLANDSVIK gave a counterexample. In this paper we state and prove a theorem similar to ZAK's but with stronger hypotheses, we show that these do not hold for Ådlandsvik's counterexample, and we point out that all of ZAK's corollaries are still implied by our version of the theorem.

0. Introduction

The extrinsic properties of an embedded projective variety X, especially concerning linear projections, are related with the properties of its secant varieties. We recall that the k-th (or the $(k-1)$-th, depending on the notation) secant variety of a given projective variety X in $\mathbf{P}^N$ is the closure in $\mathbf{P}^N$ of the union of the $(k-1)$-dimensional linear subspaces generated by k points of X.

The interest for the properties of secant varieties arose first at the beginning of the century, e.g. we can mention TERRACINI's paper [T]. In recent years, some of these properties have been used in the proof of various results in projective algebraic geometry; the most striking example is ZAK's classification of Severi varieties.

Another field of application is a new proof of linear normality for smooth low-codimension projective varieties, (i.e. such that the dimension is bigger or equal to twice the codimension); this result is important as a first step in the direction of Hartshorne's conjecture (see [H]) that low

(*) Texte reçu le 25 juillet 1989, révisé le 5 février 1990.
B. FANTECHI, Scuola Normale Superiore di Pisa, Piazza Cavalieri, 56100 Pisa, Italie.

codimension varieties should be complete intersections.

In his paper [Z2, p.168] ZAK stated a theorem of superadditivity for secant defects of projective varieties. The "expected" dimension of the k-th secant variety is equal to 1 plus the dimension of X plus the dimension of the $(k-1)$-th secant variety, as long as the latter is not all of $\mathbf{P}^N$. The difference beween the actual dimension and the expected dimension for the k-th secant variety is called the k-th secant defect, and denoted by δ_k for short. (For more precise definitions, we refer the reader to paragraph 1.) ZAK deduced from his theorem another proof of (corollary, p. 170) linear normality for smooth low-codimension varieties, and more generally he gave estimates for the possible dimension of a nondegenerate embedding for a smooth, projective variety of given dimension and first secant defect.

Subsequently, Ådlandsvik ([Å2], see also remark 3.6) gave a counterexample to ZAK's theorem, thus leaving open the question whether or not his other results were correct.

In this paper we introduce the notion of almost smooth variety; it is a projective variety X such that, for every x in X, its tangent star at x (the union of the limits of secants with endpoints tending to x) is contained in the closure of the union of the secants through x (*cf.* section 2). In general the tangent star contains the tangent cone and is contained in the Zariski tangent space; hence, in particular, every smooth variety is almost smooth.

Following the ideas of ZAK, we prove (for notations see section 1) :

THEOREM 2.5. — *Let X be an irreducible, closed subvariety of $\mathbf{P}^N$ and let k, l, m be positive integers, with $l+m=k$, $k \le k_0(X)$ and $2m \le k_0$. Assume $J^m(X)$ is almost smooth. Then $\delta_k \ge \delta_l + \delta_m$.*

Although our statement is a slightly weaker version of ZAK's theorem we remark (3.7) that it is strong enough to ensure the validity of all the main results of ZAK's paper, and in particular the linear normality result, and the estimates of embedding dimensions. We also point out that the extra hypothesis needed to make the proof work fails in fact for Ådlandsvik's counterexample.

This paper goes as follows : in section 1 we recall briefly definitions and properties of secant varieties which will be used in the sequel; we also introduce the necessary notation; in section 2 we state the main theorem and give an outline of the proof and in section 3 we deal with some technical lemmas, we describe Ådlandsvik's counterexample and we recall briefly how ZAK derived the linear normality result and the embedding dimension estimates.

I would like to thank Fabrizio CATANESE, who introduced me to this subject and helped me during my thesis work.

1. Definitions and basic properties of secant varieties

We shall denote by $\mathbf{P}^N$ the N-dimensional projective space over a fixed algebraically closed field k. If $X_1, \ldots, X_n$ are subsets of $\mathbf{P}^N$, we shall denote by $\langle X_1, \ldots, X_n \rangle$ their linear span, i.e. the smallest projective subspace of $\mathbf{P}^N$ containing all the X_j's.

Definition 1.1. — Let $X_1, \ldots, X_n$ be irreducible subvarieties of $\mathbf{P}^N$, and let

$$h = \max \{\dim \langle x_1, \ldots, x_n \rangle \mid x_j \in X_j\}.$$

We denote by $S(X_1, \ldots, X_n)$ the closure in $X_1 \times \cdots \times X_n \times \mathbf{P}^N$ of the set

$$(*) \qquad \Big\{(x_1, \ldots, x_n, z) \mid z \in \langle x_1, \ldots, x_n \rangle,\ \dim \langle x_1, \ldots, x_n \rangle = h\Big\}.$$

The projection of $S(X_1, \ldots, X_n)$ in $\mathbf{P}^N$ will be called the *join* of $X_1, \ldots, X_n$ and will be denoted by $J(X_1, \ldots, X_n)$.

Remark 1.2. — *Let $X_1, \ldots, X_n$ be irreducible subvarieties of $\mathbf{P}^N$. Then the following hold :*

(i) *$S(X_1, \ldots, X_n)$ is irreducible ;*
(ii) *$J(X_1, \ldots, X_n)$ is irreducible ;*
(iii) $\dim S(X_1, \ldots, X_n) = h + \sum \dim X_j$ *;*
(iv) *If $X_1 \cup X_2$ contains at least two distinct points*

$$\dim S(X_1, X_2) = \dim X_1 + \dim X_2 + 1.$$

Proof. — Let π denote the natural map from $S(X_1, \ldots, X_n)$ to $X_1 \times \cdots \times X_n$.

(i) and (iii) both follow by observing that the open set defined in (*) is a projective bundle of rank h over the set $\{(x_1, \ldots, x_n) \mid \dim \langle x_1, \ldots, x_n \rangle = h\}$, which is open in $X \times \cdots \times X$.

(ii) is a consequence of (i), and (iv) is a consequence of (iii). □

Definition 1.3. — $J(X_1, \ldots, X_n)$ is said to be *nondegenerate* if

$$\dim J(X_1, \ldots, X_n) = \dim S(X_1, \ldots, X_n).$$

LEMMA 1.4 (TERRACINI). — *Let X, Y be irreducible subvarieties of $\mathbf{P}^N$, $x \in X$, $y \in Y$, $z \in \langle x, y \rangle$. Then the following hold :*

(i) $T_{X,x} \subset T_{J(X,Y),z}$; $T_{Y,y} \subset T_{J(X,Y),z}$.

(ii) *If char* $k = 0$, *there exists* U *open subset of* $J(X,Y)$ *such that for all* $z \in U$, *for all* $x \in X$, *for all* $y \in Y$ *such that* $z \in \langle x, y \rangle$ *we have* $\langle T_{X,x}, T_{Y,y} \rangle = T_{J(X,Y),z}$.

Proof. — We can clearly restrict ourselves to an affine space $\mathbf{A}^N$. We shall use the same symbol to denote both a projective variety and its intersection with $\mathbf{A}^N$. Then we can define a dominant map φ : $X \times Y \times \mathbf{A}^1 \to J(X,Y)$ by $\varphi(x,y,\lambda) = \lambda x + (1-\lambda)y$. Now it is enough to show that, for $z \neq x$ and $z \neq y$, the image at (x,y,λ) of the tangent space of $X \times Y \times \mathbf{A}^1$ via $d\varphi$ is $\langle T_{X,x}, T_{Y,y} \rangle$; in fact (i) follows immediately, and (ii) follows by remarking that, in characteristic zero, the differential of a dominant map is surjective. Let $z = \varphi(x,y,\lambda)$, $T_{X,x} = V + x$, $T_{Y,y} = W + y$ with V, W vector subspaces of $\mathbf{A}^N$; then

$$d\varphi(T_{X\times Y\times \mathbf{A}^1,(x,y,\lambda)}) = \Big\{\lambda(v+x) + (1-\lambda)(w+y) + \mu(x-y) \mid \text{such that } v \in V,\ w \in W,\ \mu \in T_{\mathbf{A}^1,\lambda} = \mathbf{A}^1\Big\}.$$

It is easy to verify that this is exactly $\langle T_{X,x}, T_{Y,y} \rangle$. □

TERRACINI's proof can be found in [T]. This lemma can be easily extended to the case of $J(X_1, \dots, X_n)$, for any n.

Definition 1.5. — Let X in $\mathbf{P}^N$ be an irreducible variety. We denote $S(X, \dots, X)$ (k copies) by $S^k(X)$; in the same way we denote $J(X, \dots, X)$ by $J^k(X)$. $J^k(X)$ will be called *k-th secant variety* of X.

Notation 1.6. — We shall denote by $s_k(X)$, or s_k, the dimension of the k-th secant variety; we shall denote by $k_0(X)$, or k_0, the biggest k such that $s_k < N$.

Definition 1.7. — If $a_1, \dots, a_r$ are integers ≥ 1 such that $\sum a_i = k$, clearly

$$J\big(J^{a_1}(X), \dots, J^{a_r}(X)\big) = J^k(X).$$

We define $S^{a_1,\dots,a_r}(X)$ to be $S(J^{a_1}(X), \dots, J^{a_r}(X))$.

We now want to choose in the varieties we just defined some "good" open sets, where we shall be able in the following to construct explicitly some useful maps.

Notation 1.8. — Let $\tilde{S}^k(X)$ be the open set in $S^k(X)$ defined as follows :

$$\tilde{S}^k(X) = \Big\{(x_1, \dots, x_k, u) \text{ such that } x_i \in X,\ u \in \langle x_1, \dots, x_k \rangle,\ u \notin J^{k-1}(X)\Big\}$$