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PERIOD MAPPING VIA BRIESKORN MODULES

BY

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RÉSUMÉ. — Pour une déformation μ-constante de fonctions holomorphes à singularités isolées, on définit une application de période qui associe le module de Brieskorn à chaque point, où on utilise le système de Gauss-Manin pour définir la translation parallèle. On démontre qu'elle induit un morphisme fini de la strate μ-constante d'une déformation miniverselle. Cela signfie que le module local de fonction est déterminé par le module de Brieskorn à une ambiguité finie près.

ABSTRACT. — For a μ-constant deformation of holomorphic functions with isolated singularities, we define a period mapping by associating the Brieskorn module to each point, where we use the Gauss-Manin system to define the parallel translation. We show that it induces a finite morphism of the μ-constant stratum of a miniversal deformation. This means that the local moduli of function is determined by Brieskorn module up to finite ambiguity.

Introduction

Let $f : (\mathbb{C}^n, 0) \times (S, 0) \to (\mathbb{C}, 0)$ be a μ-constant deformation of holomorphic function with isolated singularity at $0 \in \mathbb{C}^n$, parametrized by a complex analytic space $(S, 0)$. Let f_s be the restriction of f to $(\mathbb{C}^n, 0) \times \{s\}$ for $s \in S$. By [26] we have a canonical mixed Hodge structure on the cohomology of the Milnor fiber of f_s. Then we can define a period mapping, assuming S contractible (by shrinking S if necessary). The weight filtration of the mixed Hodge structure is determined by the monodromy (so called the monodromy filtration) and remains constant by μ-constant deformation, and only the Hodge filtration varies.

Here we use Deligne's vanishing cycle sheaf [6] along f, which enables us to avoid a delicate problem about the topological triviality of μ-constant deformation. This is a locally constant sheaf of vanishing cohomologies (up to a shift) on $\{0\} \times S$, and induces the parallel translation of the Hodge

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filtration which is needed to define the period mapping. We can show also that the mapping is holomorphic (i.e., the Hodge filtration determines holomorphic vector subbundles of the vector bundle corresponding to the local system of vanishing cohomologies), even if S is singular. Note that the notion of variation of (mixed) Hodge structure is not yet defined on singular spaces.

Unfortunately this period mapping does not provide enough information, because the fibers of the mapping have positive dimensions in general (e.g., $f_0 = x^5 + y^4$), even when S is the μ-constant stratum of the base space of a miniversal deformation of f_0, cf. also (3.4). So we consider a refinement of the period mapping using *Brieskorn module.* For $s \in S$, Brieskorn module of f_s is defined by :

$$\mathcal{H}_s'' = \Omega^n_{\mathbf{C}^n,0}/\mathrm{d}f_s \wedge \mathrm{d}\Omega^{n-2}_{\mathbf{C}^n,0}$$

[3] which has the structure of $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-module [13], [17], i.e. $\mathbb{C}\{t\}$-module with (regular) singular connection ∇ such that the inverse of $\partial_t = \nabla_{\partial/\partial_t}$ is well-defined, cf. [3], where t is the coordinate of $\mathbb{C}$. Let $\mathcal{G}_s$ be the localization of $\mathcal{H}_s''$ by the action of ∂_t^{-1}. It is a regular holonomic $\mathcal{D}_{\mathbb{C},0}$-module, called the *Gauss-Manin system* of f_s [17], and $\mathcal{H}_s''$ is a $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodule of $\mathcal{G}_s$. By VARCHENKO [27] (see also [17], [20], [21], [24]), the mixed Hodge structure on the cohomology of Milnor fiber can be obtained by taking Gr_V of $\mathcal{H}_s''$, cf. (2.6.1), where V is the filtration of $\mathcal{G}_s$ by eigenvalues of the action of $\partial_t t$.

So the Brieskorn module gives finer information than the mixed Hodge structure. Using Deligne's vanishing cycle sheaf again, we show that the $\mathcal{G}_s$ ($s \in S$) form a locally constant sheaf of regular holonomic $\mathcal{D}_{\mathbb{C},0}$-modules on S, and get the parallel translation of the elements of $\mathcal{G}_s$, cf. (2.9). Assume S contractible so that $\mathcal{G}_s$ for $s \in S$ is identified with each other by parallel translation, and denote $\mathcal{G}_s$ by $\mathcal{G}$. Then the $\mathcal{H}_s''$ are $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodules of $\mathcal{G}$ parametrized holomorphically by S, i.e., $\mathcal{H}_s''$ ($s \in S$) determines a locally free subsheaf of the holomorphic scalar extension of the above locally constant sheaf (cf. (2.7-8) for a precise statement). So we get a refined period mapping :

$$\Psi : S \to \mathbf{L}(\mathcal{G}) \tag{0.1}$$

by associating Brieskorn module $\mathcal{H}_s''$ to $s \in S$, where $\mathbf{L}(\mathcal{G})$ is a set of $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$-submodules of $\mathcal{G}$ satisfying some conditions, cf. (2.9). Here the locally constant sheaf of regular holonomic $\mathcal{D}_{\mathbb{C},0}$-modules plays the role of the local system of vanishing cycles in the period mapping via the Hodge filtration on the vanishing cohomology. The problem is whether (0.1) is injective when S is the μ-constant stratum of the base space of a miniversal deformation of f_0, i.e. :

(0.2) *Problem : Is the local moduli of μ-constant deformation determined completely by Brieskorn modules ?*

See Supplement of [20]. In [22, 2.10] we proved :

THEOREM 0.3. — (0.1) *is (locally) injective on the smooth points of the μ-constant stratum*, cf. (3.2).

In this paper we show in general :

THEOREM 0.4. — (0.1) *is finite to one by shrinking the μ-constant stratum S if necessary*, cf. (3.3).

The proof is not difficult, once the period mapping is defined. In fact, since Ψ is analytic, (0.4) is reduced to $\dim \Psi^{-1}\Psi(0) = 0$, and follows from (0.3), restricting to the smooth points of $\Psi^{-1}\Psi(0)$ if $\dim \Psi^{-1}\Psi(0) > 0$. So the local moduli is determined up to finite ambiguity. For the moment, I don't have enough evidence to conjecture the injectivity or non injectivity of Ψ.

As a corollary of (0.4), we can get some information about the failure of the injectivity of the period mapping via mixed Hodge structures as above, cf. (3.4), using the structure of Brieskorn module [22], because the Hodge filtration of the vanishing cohomology is obtained by the graduation of the Brieskorn module by the filtration V, cf. (2.6.1), where the information lost by the graduation is expressed by the linear mappings $c_{\beta,\alpha}$ in [*loc. cit.*]. But it is not easy to relate this directly with the geometry of the discriminant as in [8].

Note that the mixed Hodge structure and the associated period mapping in [*loc. cit.*] are not well-defined because he considers deformation of hypersurface instead of function. Although there is an embedding of the base space of the miniversal deformation of hypersurface into the product of an open disc with the base space of the miniversal deformation of function, it is not unique and the period mapping of the μ-constant stratum obtained by composition is not well-defined. It is not clear whether we can get an interesting result using a not well-defined mapping. Note also that the theory of logarithmic vector fields and primitive forms are not so useful for the study of the Torelli problem as in [*loc. cit.*], because the

Euler vector field E is everywhere non zero (i.e. $\mathrm{Fix}(E) = \emptyset$) and the variation of mixed Hodge structure is not constant on the logarithmic strata contained in the μ-constant stratum in general, cf. (3.5).

We also give a counter example to [8, (5.3)], cf. (3.9). It is possible to give a correct (but rather transcendental) version of [*loc. cit.*] using MALGRANGE's extension of a good basis [31], cf. (3.10).

In § 1, we review some elementary facts from the theory of regular holonomic $\mathcal{D}$-module of one variable. Using this, we construct the period mapping Ψ of a not necessary smooth base space in § 2. Then the main theorems are proved in § 3.

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1. Regular holonomic $\mathcal{D}$-modules of one variable

We review some facts from the elementary theory of regular holonomic $\mathcal{D}$-module of one variable which will be needed in the next section, see also [1], [2], [7], [22, § 1], etc.

1.1. — We denote by Δ an open disc $\{t \in \mathbb{C} : |t| < \delta\}$. Put :

$$\mathcal{O} := \mathcal{O}_{\Delta,0} = \mathbb{C}\{t\}, \quad \mathcal{D} := \mathcal{D}_{\Delta,0} = \mathbb{C}\{t\}[\partial_t].$$

Let $M_{\mathrm{rh}}(\mathcal{D})$ be the category of regular holonomic $\mathcal{D}$-modules, i.e. $\mathcal{D}$-modules M of finite type such that $M[t^{-1}]$ (localization by t) are $\mathbb{C}\{t\}[t^{-1}]$-modules of finite type with regular singular connection in the classical sense, cf. [7]. Let $M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0$ be the category of regular holonomic $\mathcal{D}_\Delta$-Modules whose characteristic varieties are contained in $T_0^*\Delta$ (i.e. their stalks at 0 belong to $M_{\mathrm{rh}}(\mathcal{D})$ and their restrictions to the punctured disc Δ^* are locally free finite $\mathcal{O}_{\Delta^*}$-Modules with integrable connection). Then $M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0$ is independent of Δ by restriction morphisms, because the locally free finite $\mathcal{O}_{\Delta^*}$-Module with integrable connection can be uniquely extended to a larger punctured disc. So we have an equivalence of categories :

$$M_{\mathrm{rh}}(\mathcal{D}_\Delta)_0 \xrightarrow{\sim} M_{\mathrm{rh}}(\mathcal{D}) \tag{1.1.1}$$

using coherent extension of $\mathcal{D}$-modules. For $M \in M_{\mathrm{rh}}(\mathcal{D})$ and $\alpha \in \mathbb{C}$, let :

$$M^\alpha = \bigcup_{i \geq 0} \mathrm{Ker}\big((\partial_t t - \alpha)^i : M \to M\big). \tag{1.1.2}$$