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Bernard Host
Jean-François Méla
François Parreau

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NON SINGULAR TRANSFORMATIONS AND SPECTRAL ANALYSIS OF MEASURES

BY

BERNARD HOST, JEAN-FRANÇOIS MÉLA

AND FRANÇOIS PARREAU (*)

RÉSUMÉ. — Ce travail approfondit les interactions qui existent entre l'analyse harmonique des mesures et l'étude spectrale des systèmes dynamiques non singuliers. Il est centré sur l'étude de sous-groupes remarquables du cercle, groupes de valeurs propres, groupes de quasi-invariance des mesures..., dont les exemples les plus naturels sont définis par des conditions diophantiennes. La conjonction des points de vue permet d'obtenir nombre de résultats nouveaux dans les deux théories, y compris dans des problèmes classique d'analyse de Fourier.

ABSTRACT. — This work explores in depth the interactions existing between harmonic analysis of measures and spectral theory of non-singular dynamical systems. It focuses on the study of some classes of remarkable subgroups of the circle : eigenvalue groups, groups of quasi-invariance of measures..., the most natural examples of which are defined by diophantine conditions. The conjonction of these points of view leads to many new results in both theories, including some classical problems in Fourier analysis.

1. Introduction

The spectral study of non-singular transformations reveals a deep interplay between ergodic theory and harmonic analysis. The aim of this work is to display some aspects of these connections, mainly those involving the *eigenvalue group* $e(T)$ of a non-singular transformation T, and the *group of quasi-invariance* $H(\mu)$ of a positive finite Borel measure μ on $\mathbb{T}$, that is the group of all $t \in \mathbb{T}$ such that μ is equivalent to its translate by t.

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B. HOST, J.-F. MÉLA, F. PARREAU, Université Paris Nord, C.S.P., Dépt. de Mathématiques, Laboratoire d'Analyse et Applications, URA 742 du CNRS, avenue J.B. Clément, 93430 Villetaneuse, France.

The work finds its unity as much in the methods as in the results, as the reader will realize, we hope.

For sake of simplicity, we will focus on the case of the circle group, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is maybe the most interesting one. Most of our results can be easily extended to any second countable locally compact group and we will mention it only occasionally. We use the classical notations and results of Ergodic Theory and Fourier Analysis. The measure spaces we consider are all standard, that is : up to an isomorphism, they are polish spaces with a finite or σ-finite positive Borel measure. In the case of $\mathbb{T}$, we also allow complex Borel measures.

Along this paper, we will introduce some remarkable Borel subgroups of $\mathbb{T}$ which all belong to the class of so-called *saturated subgroups* of $\mathbb{T}$.

The notion of saturated subgroup had already been considered in a more general setting and under an other name, in the chapter 8 of [18]. The point was there to exhibit some examples of non locally compact group topologies for which we have an extension of the Bochner theorem. The property for a subgroup to be saturated is initially a Fourier Analysis property, related to the notions of *Dirichlet measure* and *weak Dirichlet set* ([18], [26]). C. MOORE and K. SCHMIDT ([32], [41]) already noticed that these notions appear naturally when studying the eigenvalue groups of non-singular dynamical systems.

Since then, there have been some developments and it seems useful to give a clear, new and more complete exposition of these topics. This is achieved in section 2, where are also discussed the connections with the classical theory of the absolute convergence of trigonometric series. We introduce a group which plays a key role at the crossroad of harmonic analysis and non-singular dynamics. Given any positive finite Borel measure on $\mathbb{T}$, we denote by $\overline{\mathbb{Z}}_1(\mu)$ the group of all measurable functions (of unit modulus) which are limits of group characters $e^{2\pi i n x}$, $n \in \mathbb{Z}$, in the $L^1(\mu)$ or $L^2(\mu)$ topology. We also characterize saturated subgroups in terms of these groups.

In the same section, we define a class of subgroups which provides typical examples throughout the paper : with any sequence of positive integers n_j and any sequence of nonnegative real numbers a_j such that $\sum a_j = +\infty$, we associate the group of all $t \in \mathbb{T}$ such that

$$\sum_{0}^{+\infty} a_j \left| e^{2\pi i n_j t} - 1 \right|^2 < +\infty.$$

Such a group will be called an *H_2 group*.

In section 3, we study the eigenvalue groups of ergodic non-singular transformations or flows, which have been paid attention by several authors ([14], [19], [3]) and which are also the T-sets of the Connes-Krieger theory of Von-Neumann algebras ([9]). These groups may be uncountable but may not be any subgroup of the circle; they are σ-compact and admit an intrinsic polish topology stronger than the circle topology. C. MOORE and K. SCHMIDT ([32], [41]) first noticed that eigenvalue groups are weak Dirichlet sets, except when equal to $\mathbb{T}$.

In fact it is proved in [30] that they enjoy the stronger property to be saturated. However, this proof is a little sketchy and the properties of saturated subgroups are simply stated, referring to [18] where the terminology and the context are different. The main theorem of [30] is proved here with full details and complements. Moreover, having in mind some general applications in Harmonic Analysis, we get rid of the unnecessarily restrictive assumption of ergodicity (restricting then the definitions to eigenfunctions of constant modulus).

We ask wether the conjunction of the saturation property and of the topological properties of eigenvalue groups characterizes this class of subgroups of $\mathbb{T}$. We prove that any H_2 group can be realized as the eigenvalue group of some ergodic non-singular transformation (3.5). Besides, any σ-compact saturated subgroup of the circle is close to being an H_2 group (2.3); it might even be that every eigenvalue group is an H_2 group.

The main proof in section 3 involves the construction of factors which play the same role as discrete factors for measure preserving transformations. Denoting by S the transformation of $\overline{\mathbb{Z}}_1(\tau)$ corresponding to the multiplication by $e^{2\pi i x}$, any non-singular system $(\overline{\mathbb{Z}}_1(\tau), \nu, S)$ is isomorphic to a non-singular compact group rotation (THEOREM 3.2). Given a non-singular dynamical system (X, μ, T), we associate such a factor with every positive Borel measure τ carried by $e(T)$.

In section 4, we consider some classical non-singular systems defined as Kakutani towers over the 2-odometer, which appear in [10], [23] and are also studied in [33], [19], [3]. Their eigenvalue groups turn out to be H_2 groups. We will prove two new results for these systems. First, under some growth condition for the height function, we show that the tower is isomorphic to a non-singular compact group rotation : we construct on the eigenvalue group a probability measure which is familiar to harmonic analysts — a so called "generalized Riesz product" — and then we prove that the factor given by the technique of section 3 is in fact isomorphic to the initial system.

On the other hand, in full generality, we compute the maximal spectral types of these systems which turn out to be classical Riesz products, and it is worth noting that any Riesz product with nonnegative coefficients may be interpreted as a maximal spectral type in this way.

Both results point out the key role of Riesz products in these topics and provide a link with the last sections of the paper, where we are mainly concerned with groups of quasi-invariance of measures on the circle.

Indeed, it is known that for any non-singular transformation T with maximal spectral type σ the eigenvalue group $e(T)$ is contained in the quasi-invariance group $H(\sigma)$. Whether the equality holds in general is an open problem. In the case of the tower over the 2-odometer and the Riesz product, we are able to prove it under a mild condition on the height function. This yields a condition for the equivalence of a Riesz product with its translates which improves the classical results (G. BROWN and W. MORAN [6], J. PEYRIÈRE [37]). Besides, when the tower is isomorphic to a non-singular compact group rotation, its spectral type is ergodic under the action of some countable group of translations. This provides a proof of the ergodicity of a class of Riesz products by a method quite different from PARREAU's [34].

Apart from these results, the section 5 contains a general study of the groups $H(\mu)$ for an arbitrary positive measure μ on $\mathbb{T}$. Till recently, very little was known about these groups, except that $\mu(H(\mu)) = 0$ for any continuous singular measure μ ([29] ; see also [12]). J. AARONSON and M. NADKARNI [3] show that $H(\mu)$ is the eigenvalue group of some non-singular transformation and thus is a saturated subgroup, under the assumption of ergodicity which is not quite natural from the Harmonic Analysis point of view. We prove the result without restriction, as a particular case of a more general theorem which deals with cocycle extension (THEOREM 5.4) : the "maximal group" for a cocycle associated with a group of translations on $\mathbb{T}$ is an eigenvalue group. This contains some results by H. HELSON and K. MERRILL ([16], [17]).

The property for a measure on $\mathbb{T}$ to be "ergodic under translations" may be restated in a non-classical way : it is possible to drop any requirement of quasi-invariance (see [7], [12]) and moreover any reference to a specified group of translations [34]. In this context the natural object to attach to any positive finite Borel measure μ on $\mathbb{T}$ is the set $A(\mu)$ of all $t \in \mathbb{T}$ such that μ and its translate by t are not mutually singular. These topics are discussed in section 5.5, where we also prove a significant property of the sets $A(\mu)$ for an arbitrary singular measure μ : any measure carried by $A(\mu)$, although it need not be a