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CAUCHY-FANTAPPIÈ-LERAY FORMULAS WITH LOCAL SECTIONS AND THE INVERSE FANTAPPIÈ TRANSFORM

BY

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RÉSUMÉ. — Nous déduisons une formule du type Cauchy-Fantappiè-Leray n'utilisant que des sections localement définies. A l'aide de cela, nous construisons une formule d'inversion pour la transformation de Fantappiè dans le cas $\mathbb{C}$-convexe général. Ceci rétablit la moitié non démontrée d'une conjecture de Aïzenberg, Trutnev et Znamenskij affirmant qu'un domaine est $\mathbb{C}$-convexe si et seulement si la transformation de Fantappiè y est un isomorphisme.

ABSTRACT. — We derive a Cauchy-Fantappiè-Leray formula that requires only locally defined sections. We use it to construct an inversion formula for the Fantappiè transform in the general $\mathbb{C}$-convex case. This establishes the unproved half of a conjecture of Aizenberg, Trutnev and Znamenskij that states that a domain is $\mathbb{C}$-convex if and only if the Fantappiè transform is an isomorphism.

0. Introduction

If D is a domain in $\mathbb{C}^n$ (or $\mathbb{P}^n$) then the Fantappiè transform F maps $H'(D)$ into $H(D^*)$, where D^* is the set of all hyperplanes not intersecting D (for exact definitions see § 3). If D is convex, then F is an isomorphism. This was proved by MARTINEAU, see [2]. In the seventies AIZENBERG and TRUTNEV conjectured :

THEOREM 0. — *The Fantappiè transform F is an isomorphism if and only if D has simply connected intersections with all complex lines.*

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We call such a domain D $\mathbb{C}$-*convex.* This conjecture was announced by ZNAMENSKIJ in [8] to be true, and it is subject to a more elaborated treatment in [9], where the necessity of $\mathbb{C}$-convexity is established. However, the proposed arguments for the sufficiency are long and involved and seem to have gaps.

The aim of this paper is to give a rigorous and comprehensive proof of the sufficiency, i.e. that F is an isomorphism if D is $\mathbb{C}$-convex. Along the way we obtain a quite explicit inversion formula for F. This is constructed by means of a Cauchy-Fantappiè-Leray representation formula for holomorphic functions, in which the Cauchy-Leray sections are defined only locally. We derive the representation formula in § 1 and use it in § 2 to give a simple Hahn-Banach proof of a (known) Runge type theorem which we need later on. In § 3 we construct the inversion formula for the Fantappiè transform F when D is $\mathbb{C}$-convex, and prove that F is an isomorphism. However, the proof requires some topological facts about $\mathbb{C}$-convex domains, some of which we have not found in the literature, and we collect them with proofs in an appendix. For instance, we prove (although it is not needed in our proof of THEOREM 1) that a $\mathbb{C}$-convex domain is contractible.

We conclude this paragraph by suggesting the idea behind the Cauchy-Fantappiè representation formula. Suppose K is a compact set in $\mathbb{C}^n$ and $f \in H(\Omega)$ where $\Omega \supset\supset K$. Then there is an open set ω with smooth boundary such that $K \subset \omega \subset\subset \Omega$. If $n = 1$ we can represent f on K by the Cauchy formula :

$$f(z) = \frac{1}{2\pi i}\int_{\partial\omega}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}, \quad z \in K. \tag{1}$$

A simple generalization of (1) to $\mathbb{C}^n$ is the Bohner-Martinelli formula

$$f(z) = c_n\int_{\partial\omega}\frac{\sum(\bar\zeta_j - \bar z_j)\,\mathrm{d}\zeta \wedge \big(\sum \mathrm{d}\bar\zeta_j \wedge \mathrm{d}\zeta_j\big)^{n-1}}{|\zeta - z|^{2n}}f(\zeta), \quad z \in K, \tag{2}$$

which however has the disadvantage that the kernel is not holomorphic in z, which is crucial in certain applications.

In order to obtain a representation formula with holomorphic kernel one, roughly speaking, has to find a complex hypersurface, not intersecting K, through each point $\zeta \in \partial\omega$, and moreover do this in a C^1-manner on $\partial\omega$. More explicitly, if we have a mapping $s(\zeta, z) : \partial\omega\times\Omega \to \mathbb{C}^n$ which is holomorphic in z and such that the hypersurfaces $\{z\,;\ s(\zeta,z)\cdot(\zeta - z) = 0\}$ do not intersect K, then the Cauchy-Fantappiè-Leray formula (see § 1)

$$f(z) = c_n\int_{\partial\omega}\frac{s(\zeta,z)\wedge\big(\bar\partial s(\zeta,z)\big)^{n-1}}{\big(s(\zeta,z)\cdot(\zeta - z)\big)^{n}}f(\zeta), \quad z \in K, \tag{3}$$

holds, where $s(\zeta, z)$ is identified with the form $\sum s_j(\zeta, z)\,\mathrm{d}\zeta_j$. We will present a method for constructing a global representation formula of this kind from local (on $\partial\omega$) choices of $s(\zeta, z)$. The resulting formulas will inherit some properties of $s(\zeta, z)$, e.g. being holomorphic or algebraic in z. In case of domains with piecewise smooth boundary, our formulas are connected to Norguet's formula, see the last remark in § 1.

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1. Cauchy-Fantappiè-Leray formulas with locally defined sections

We present the formulas in $\mathbb{P}^n$-formalism since this is most natural when applied to the Fantappiè transform in § 3. However, there is a simple way to translate to the $\mathbb{C}^n$-form (see below).

Let $\zeta = (\zeta_0, \dots, \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ be homogeneous coordinates for the point $[\zeta]$ in $\mathbb{P}^n$ and let π denote the natural projection :

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n.$$

Sometimes we abusively write $\zeta \in \mathbb{P}^n$ rather than $[\zeta] \in \mathbb{P}^n$. Via the natural pairing $\langle\ ,\ \rangle$ of $\mathbb{C}^{n+1}$ and its dual $(\mathbb{C}^{n+1})^*$, the elements in $(\mathbb{P}^n)^* = \big\{[\xi]\,;\ \xi \in (\mathbb{C}^{n+1})^* \setminus \{0\}\big\}$ are identified with the hyperplanes in $\mathbb{P}^n$, i.e. $[\xi] \sim \big\{[\zeta] \in \{[\zeta] \in \mathbb{P}^n\,;\ \langle \zeta, \xi\rangle = 0\big\}$ and vice versa.

A fixed choice of hyperplane $\eta^* \in (\mathbb{P}^n)^*$ in $\mathbb{P}^n$ (called the *hyperplane at infinity*) defines a unique affine structure on $\mathbb{P}^n \setminus \eta^*$, making it affine-isomorphic to $\mathbb{C}^n$.

If Ω is an open set in $\mathbb{P}^n$ there is a 1-1 correspondence between holomorphic functions in Ω, $H(\Omega)$, and zero-homogeneous holomorphic functions in $\pi^{-1}\Omega \subset \mathbb{C}^{n+1} \setminus \{0\}$. Let

$$X = \big\{([\zeta], [\xi]) \in \mathbb{P}^n \times (\mathbb{P}^n)^*\,;\ \langle \zeta, \xi\rangle = 0\big\}$$

and $c_n = (2\pi i)^{-n}$.

PROPOSITION 1. — *If $F(\zeta)$ and $\Phi(\xi)$ are $-n$-homogeneous and holomorphic in (some open subsets of) $\mathbb{C}^{n+1} \setminus \{0\}$ and $(\mathbb{C}^{n+1})^* \setminus \{0\}$, respectively, then*

$$\alpha(\xi, \zeta) = c_n \Phi(\xi) \sum_0^n \xi_j\,\mathrm{d}\zeta_j \wedge \Big(\sum_0^n \mathrm{d}\xi_j \wedge \mathrm{d}\zeta_j\Big)^{n-1} F(\zeta) \tag{1}$$

is a well-defined closed form on (some open subset of) X.

Proof. — If for instance $\zeta_0 \neq 0$ and $\xi_0 \neq 0$, then

$$\alpha(\xi,\zeta) = c_n \xi_0^n \Phi(\xi) \sum_1^n \frac{\xi_j}{\xi_0} \mathrm{d}\frac{\zeta_j}{\zeta_0} \wedge \Big(\sum_1^n \mathrm{d}\frac{\xi_j}{\xi_0} \wedge \mathrm{d}\frac{\zeta_j}{\zeta_0}\Big)^{n-1} \zeta_0^n F(\zeta) \tag{2}$$

which shows that α indeed is well-defined on $\mathbb{P}^n \times (\mathbb{P}^n)^*$ and in particular on the submanifold X. Since $F(\zeta)$ is holomorphic it follows of bidegree reasons that $\mathrm{d}_\zeta \alpha = 0$, but since $\sum \xi_j \,\mathrm{d}\zeta_j = -\sum \zeta_j \,\mathrm{d}\xi_j$ on X and also $\Phi(\xi)$ is holomorphic, we similarly have $\mathrm{d}_\xi \alpha = 0$ on X. □

Definition. — A CL-*section* (Cauchy-Leray section) $s(\zeta)$ is a C^1-mapping from some subset of $\mathbb{P}^n$ into $(\mathbb{P}^n)^*$ such that $\langle \zeta, s(\zeta)\rangle = 0$, i.e. such that $s(\zeta)$ is a hyperplane through ζ.

If $\Phi(\xi)$ is holomorphic on the image of $s(\zeta)$ it follows from PROPOSITION 1 that $\alpha(s(\zeta),\zeta)$ is a closed $(n,n-1)$-form in ζ.

PROPOSITION 2 (The Cauchy-Fantappiè-Leray formula). — *Suppose that $\omega \subset\subset \mathbb{P}^n \setminus \eta^*$ has smooth boundary and that $f \in H(\bar\omega)$. If $z \in \omega$ and $s(\zeta)$ is a* CL-*section over $\partial\omega$ such that $s(\zeta)$ does not contain z, then*

$$f(z) = c_n \int_{\partial\omega} \frac{\langle z,\eta^*\rangle^n}{\langle z,\xi\rangle^n}\, \xi \wedge (\mathrm{d}\xi)^{n-1}|_{\xi = s(\zeta)}\, f(\zeta)\, \frac{1}{\langle \zeta,\eta^*\rangle^n} \tag{3}$$

where ξ is identified with the form $\sum_0^n \xi_j \,\mathrm{d}\zeta_j$.

This well-known formula is a special case of THEOREM 4 below.

Remark. — If $\eta = \eta^* = (1,0,0,\ldots)$ and we make the identifications $[\zeta] = [(1,\zeta')] \sim \zeta' \in \mathbb{C}^n, [z] = [(1,z')] \sim z' \in \mathbb{C}^n$ and $[s] = [(s_0,s')] \sim s'$, noting that $s(\zeta)$ being a CL-section means that $s_0(\zeta) = -s'(\zeta)\cdot\zeta'$, then (3) becomes (3) in § 0. In particular, if $s(\zeta)$ is the complex tangent plane to the level surface of the distance function $d(\zeta,z)$ in $\mathbb{C}^n$, then (3) becomes the Bochner-Martinelli formula (2) in § 0. □

If we for each z in, say, $K \subset \omega$ have a section $\zeta \mapsto (\zeta,z)$ that does not intersect z, then of course (3) holds for all $z \in K$. Then $s(\zeta,z)$ also may have some additional desirable property; e.g. being constant, a polynomial or at least holomorphic in z. We are going to construct formulas when such a required $s(\zeta,z)$ only is found locally on $\partial\omega$. To this end, we will use the following formalism :

Let $z \in \mathbb{P}^n/\eta^*$ be fixed and let

$$\alpha(\xi,\zeta) = c_n \frac{\langle z,\eta^*\rangle^n}{\langle z,\xi\rangle^n}\, \xi \wedge (\mathrm{d}\xi)^{n-1} \frac{1}{\langle \zeta,\eta^*\rangle^n}\cdot$$