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MOTOO UCHIDA

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CONTINUATION OF ANALYTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS UP TO CONVEX CONICAL SINGULARITIES

BY

MOTOO UCHIDA (*)

RÉSUMÉ. — On étudie le prolongement analytique des solutions d'équations aux dérivées partielles linéaires jusqu'au sommet d'un cône convexe fermé. On démontre en particulier un théorème général sur les singularités apparentes des solutions analytiques réelles d'équations aux dérivées partielles à coefficients variables.

ABSTRACT. — We study analytic continuation of the solutions of linear partial differential equations up to the vertex of a closed convex cone. In particular, we prove a general theorem on removable singularities of the real analytic solutions of partial differential equation with variable coefficients.

0. Introduction

In this paper, we study problems of local continuation of real analytic solutions of differential equations in a unified manner and, in particular, prove the following theorems :

THEOREM 0.1. — *Let K be a C^1-convex closed subset of a real analytic manifold M, having a conical singularity at x (cf. section 1.1). Let $P = P(x, D)$ be a second order differential operator with analytic coefficients defined in a neighborhood of x. Assume that P is of real principal type and is not elliptic. Then any real analytic solution to the equation $Pu = 0$ defined outside K is analytically continued up to x.*

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Motoo UCHIDA, Osaka University, College of general education, Mathematics, Toyonaka, Osaka 560, Japan.
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THEOREM 0.2. — *Let (K, x) be as in the above theorem. Let $\mathcal{M}$ be an elliptic system of linear differential equations in a neighborhood of x. Assume that the characteristic variety of $\mathcal{M}$ has codimension ≥ 2 in the fibre of x. Then any (real analytic) solution of $\mathcal{M}$ defined outside K is analytically continued up to x.*

We also give a similar theorem to 0.2 for a class of overdetermined systems of linear differential equations including systems which are not elliptic. In section 6, we prove a theorem on removable isolated singularities of the real analytic solutions of higher order partial differential equations with analytic coefficients, without any growth condition (cf. COROLLARY 6.2). This theorem extends the result of [G] for C^∞-solutions of differential equations with constant coefficients to those with variable (analytic) coefficients.

As for global continuation of solutions of linear differential equations with constant coefficients, there is much literature : [E], [M], [P], [Ko], [G], [Kn5], and the references cited there. In the case of overdetermined systems of differential equations with analytic coefficients, KAWAI [Kw] has given general results on local continuation of hyperfunction/analytic function solutions. On the other hand, in the case of single differential equations with variable coefficients, KANEKO (cf. survey report [Kn1], [Kn2]) proved, by a different method, a theorem on local continuation of real analytic solutions with thin singularity; however, his argument holds good only when the singularity of an analytic solution is contained in a hypersurface. The purpose of this paper is to deal with the local continuation problem of solutions of differential equations, with sharp-pointed bulky singularity. It should be noted that our argument is closely related to that of [Kn2, part 3].

1. Main Results

1.1. Notations. — Let M be a real analytic manifold of dimension $n \geq 2$, X a complex neighborhood of M, $\pi : T^*X \to X$ the cotangent bundle of X. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, $\mathcal{A}_M$ the sheaf of real analytic functions on M (i.e., $\mathcal{A}_M = \mathcal{O}_{X\,|\,M}$).

Let $P = P(x, D)$ be a differential operator with analytic coefficients on M. We denote by ${\mathcal{A}_M}^P$ the sheaf of real analytic solutions to the equation $Pu = 0$; i.e., ${\mathcal{A}_M}^P$ denotes the kernel of the sheaf homomorphism $P : \mathcal{A}_M \to \mathcal{A}_M$.

Let K be a closed subset of M. K is said to be *C^α-convex at $x \in M$* $(1 \leq \alpha \leq \omega)$ if there exist a neighborhood U of x and an open C^α-

immersion $\phi : U \to \mathbb{R}^n$ such that $\phi(U \cap K)$ is convex in $\mathbb{R}^n$. For $x \in K$, the tangent cone $C_x(K)$ of K at x is defined in a system of local C^1-coordinates by

$$C_x(K) = \big\{v \in T_xM \mid \text{there are sequences } \{x_\nu\} \subset K \text{ and } \{a_\nu\} \subset \mathbb{R}_+ \\ \text{such that} \quad x_\nu \to x,\ a_\nu(x_\nu - x) \to v\big\}$$

(cf. e.g. [KS2]). K is said to have a *conical singularity at* x if $x \in K$ and $C_x(K)$ is a closed proper cone of T_xM or equivalently if, for a choice of local C^1-coordinates, there is a closed convex proper cone Γ of $\mathbb{R}^n$ with vertex at x containing K locally.

1.2. — Let $x \in M$. Let $P = P(x, D)$ be a second order differential operator with analytic coefficients defined in a neighborhood of x and let $f = f(z, \zeta)$ denote the principal symbol of P. We assume the following conditions :

(a.1) P is of real principal type :

$$\operatorname{Im} f_{|T_M^*X} = 0 \ \text{ and } \ \mathrm{d}_\zeta f \neq 0 \ \text{ on } \ f^{-1}(0) \cap \pi^{-1}(x) \setminus T_X^*X,$$

where d_ζ denotes the differential along the fibres of $\pi : T^*X \to X$.

(a.2) P is not elliptic at x :

$$f^{-1}(0) \cap \pi^{-1}(x) \cap T_M^*X \not\subset T_X^*X.$$

THEOREM 1. — *Let K be a closed C^1-convex subset of M, having a conical singularity at x. Let $P = P(x, D)$ be a second order differential operator with analytic coefficients, satisfying conditions* (a.1) *and* (a.2) *at x. Then*

$$\mathcal{A}_M{}^P \longrightarrow j_* j^{-1} \mathcal{A}_M{}^P \tag{1.1}$$

is an isomorphism at x, where j denotes the open embedding $M \backslash K \hookrightarrow M$; i.e., any real analytic solution to $Pu = 0$ defined on $M \backslash K$ can be continued up to x as a real analytic solution.

REMARK 1. — KANEKO [Kn4] conjectured the result of THEOREM 1 in the case of the wave equation $P = D_1^2 + \cdots + D_{n-1}^2 - D_n^2$ for $K = \big\{(x_1, x') \in \mathbb{R}^n \mid x_1 \leq -|x'|\big\}$ through the observations of loci of singularities of its solutions (cf. also [Kn1], [Kn2], and [Kn3]).

REMARK 2. — A generalization of THEOREM 1 to higher order differential equations in the case $K = \{x\}$ is given later in section 6.

1.3. — Let $\mathcal{D}_X$ denote the sheaf of rings of differential operators on X. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called a *system of linear differential equations*, and $\mathrm{Char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. In order to state a result for overdetermined systems, we first recall the notion of a virtual bicharacteristic manifold of $\mathcal{M}$.

Let $V = \mathrm{Char}(\mathcal{M})$; V^c denotes the complex conjugate of V with respect to T_M^*X. Let $p \in V \cap (T_M^*X \setminus M)$. Assume that V satisfies the following conditions at p :

(b.1) V is nonsingular at p.

(b.2) V and V^c intersect cleanly at p; i.e., $V \cap V^c$ is a smooth manifold and :

$$T_pV \cap T_pV^c = T_p(V \cap V^c).$$

(b.3) $V \cap V^c$ is regular; i.e., $\omega|_{V \cap V^c} \neq 0$, with ω being the fundamental 1-form on T^*X.

(b.4) The generalized Levi form of V has constant rank in a neighborhood of p.

Then one can find $\mathbb{C}^\times$-conic complex involutive manifolds V_1, V_2, V_3 so that :

(c.1) $V = V_1 \cap V_2 \cap V_3$;

(c.2) V_1 is regular and $V_1 = V_1^c$;

(c.3) V_2 and V_2^c intersect transversally and their intersection is regular and involutive;

(c.4) the generalized Levi form of V_3 is non degenerate

(cf. [SKK, chap. III, sect. 2.4]). The virtual bicharacteristic manifold Λ_p of $\mathcal{M}$ is by definition the real bicharacteristic manifold of $V_1 \cap V_2 \cap T_M^*X$ passing through p (cf. [SKK, chap. III, def. 2.2.7]; cf. also [Kw, p. 222]). In the theorem below, we assume :

(b.5) $\mathrm{d}\pi(T_p\Lambda_p) \neq \{0\}$,

where $T_p\Lambda_p$ denotes the tangent space of Λ_p at p, and

$$\mathrm{d}\pi : T_pT_M^*X \to T_{\pi(p)}M.$$

THEOREM 2. — *Let $x \in M$ and let K be a closed C^1-convex subset of M having a conical singularity at x. Let $\mathcal{M}$ be a system of differential equations. Assume that* $\mathrm{Char}(\mathcal{M}) \cap \pi^{-1}(x)$ *has codimension* ≥ 2 *in* $\pi^{-1}(x)$ *and that* $V = \mathrm{Char}(\mathcal{M})$ *satisfies conditions* (b.1)–(b.5) *at each point* p