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POLES OF IGUSA'S LOCAL ZETA FUNCTION AND MONODROMY

BY

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RÉSUMÉ. — Soit K une extension finie de $\mathbb{Q}_p$ et R son anneau de valuation. On associe à chaque $f \in K[x]$, avec $x = (x_1, \dots, x_n)$, la fonction zêta locale d'Igusa

$$Z(s) = \int_{R^n} |f(x)|^s |dx|,$$

qui est méromorphe sur $\mathbb{C}$. La *conjecture de monodromie* associe des valeurs propres de la monodromie (complexe) de l'hypersurface $f = 0$ aux pôles de $Z(s)$. On peut exprimer une liste de candidats-pôles de $Z(s)$ ainsi que les valeurs propres de la monodromie à l'aide de données numériques de variétés exceptionelles, associées à une résolution plongée de $f = 0$. En utilisant des relations entre ces données numériques on montre que certains candidats-pôles ne contribuent pas aux vrais pôles, ce qui entraîne une forte évidence concernant la conjecture.

ABSTRACT. — Let K be a finite extension of $\mathbb{Q}_p$ and R its valuation ring. To any $f \in K[x]$, with $x = (x_1, \dots, x_n)$, is associated Igusa's local zeta function

$$Z(s) = \int_{R^n} \big|f(x)\big|^s |dx|,$$

which is known to be meromorphic on $\mathbb{C}$. The *monodromy conjecture* relates poles of $Z(s)$ to eigenvalues of the (complex) monodromy of the hypersurface $f = 0$. Now we can express both a list of candidate-poles for $Z(s)$ and the monodromy-eigenvalues in terms of certain numerical data of exceptional varieties, associated to an embedded resolution of $f = 0$. Using relations between those numerical data we study the vanishing of bad candidate-poles for $Z(s)$ to obtain a lot of evidence for the conjecture.

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Introduction

Let K be a number field and R its ring of algebraic integers. For any maximal ideal $\mathfrak{p}$ of R, let $R_{\mathfrak{p}}$ and $K_{\mathfrak{p}}$ denote the completion of respectively R and K with respect to the $\mathfrak{p}$-adic absolute value. Let $|x|$ denote this absolute value for $x \in K_{\mathfrak{p}}$, and let q be the cardinality of the residue field $\overline{K} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. (For example if $K = \mathbb{Q}$ we have that $\mathfrak{p}$ is determined by a prime number p, then $K_{\mathfrak{p}}$ is the field of p-adic numbers $\mathbb{Q}_p$ and $\overline{K}$ is the finite field with p elements.)

Let $f(x) \in K[x]$, with $x = (x_1, \ldots, x_{n+1})$. Then *Igusa's local zeta function* of f is defined by

$$Z(s) = Z_{\mathfrak{p}}(s) = \int_{R_{\mathfrak{p}}^{n+1}} |f(x)|^s \, |\mathrm{d}x|,$$

where $|\mathrm{d}x|$ denotes the Haar measure normalized such that $R_{\mathfrak{p}}^{n+1}$ has measure one. It describes the Poincaré series

$$P(T) = \sum_{i=0}^{\infty} N_i \left(q^{-(n+1)}T\right)^i,$$

where N_i, with $i \in \mathbb{N}$, is the number of solutions of $f = 0$ in the ring $R_{\mathfrak{p}}/\mathfrak{p}^i R_{\mathfrak{p}}$, through the relation

$$Z(s) = (1 - q^s)\, P(q^{-s}) + q^s.$$

IGUSA [Ig1] proved that $Z(s)$, and therefore also $P(T)$, is a rational function of $q^{-s} = T$.

One can compute $Z(s)$ using an embedded resolution with normal crossings for $f = 0$ in $\mathbb{A}^{n+1}(\mathbb{Q}^a)$, where $\mathbb{Q}^a$ is the algebraic closure of $\mathbb{Q}$. (An explicit formula of DENEF [D1] is stated in THEOREM 1.2.) Let (X, h) be such a resolution, obtained by Hironaka's main theorem [Hi], and denote by E_i, with $i \in S$, the (reduced) irreducible components of $h^{-1}(f^{-1}\{0\})$. We associate to each E_i, $i \in S$, a pair of *numerical data* (N_i, ν_i) where N_i and $(\nu_i - 1)$ are the multiplicities of E_i in the divisor of respectively $f \circ h$ and $h^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1})$ on X.

In particular all real poles of $Z(s)$ are part of the set $\{-\nu_i/N_i \mid i \in S\}$. So determining the real poles consists in throwing away the bad candidates. Now it is striking that « most » candidate-poles are actually bad. This fact would be elucidated if the following *monodromy conjecture* is true.

CONJECTURE. — *For all except a finite number of* $\mathfrak{p}$ *we have that, if* s *is a pole of* $Z_{\mathfrak{p}}(s)$, *then* $\mathrm{e}^{2\pi i \operatorname{Re}(s)}$ *is an eigenvalue of the monodromy acting on the cohomology* (*in some dimension*) *of the Milnor fiber of* f *associated to some point of the hypersurface* $f = 0$.

We explain this more in detail. (For the concept of monodromy we refer to MILNOR [Mi]). Fix an exceptional variety E_j and set $\overset{\circ}{E}_j = E_j \setminus (\bigcup_{i \neq j} E_i.)$ For any scheme V of finite type over K let $\chi(V)$ denote the Euler-Poincaré characteristic of $V(\mathbb{C})$. Suppose that ν_j and N_j are coprime and that there is no E_i, with $i \in S \setminus \{j\}$, with $N_j \mid N_i$. The monodromy conjecture implies, for all except a finite number of prime ideals $\mathfrak{p}$, that $s = -\nu_j/N_j$ is no pole of $Z(s)$ if $\chi(\overset{\circ}{E}_j) = 0$. (We illustrate this in paragraph 2.) Now in any concrete example we have that $\chi(\overset{\circ}{E}_j) = 0$ for « most » exceptional varieties E_j.

IGUSA [Ig5] tested the monodromy conjecture for relative invariants of certain reductive groups. LOESER verified it for arbitrary polynomials in two variables [L1], and for polynomials which are non-degenerate with respect to their Newton polyhedron, assuming certain additional conditions [L3]. We should also mention that the archimedean analogon of the conjecture has been proved by MALGRANGE [Ma1], [Ma2].

In this paper we are interested in the vanishing of bad candidate-poles for $Z(s)$ to obtain more evidence for the monodromy conjecture, using relations between the numerical data of the resolution (X, h) for $f = 0$. Considering the formula for $Z(s)$ of THEOREM 1.2, it is clear that relations between the numerical data of E_j and of the E_i, $i \in S \setminus \{j\}$, that intersect E_j are very useful to make conclusions about the residue of $-\nu_j/N_j$.

Relations. — In [V2] we proved for arbitrary polynomials f relations between numerical data, which we state briefly in paragraph 0. We now explain the essential aspects of those relations.

Fix one exceptional variety E with numerical data (N, ν). The variety E in the final resolution X is in fact obtained by a finite succession of blowing-ups

$$E^0 \xleftarrow{\pi_0} E^1 \xleftarrow{\pi_1} \cdots E^i \xleftarrow{\pi_i} E^{i+1} \cdots \xleftarrow{\pi_{m-2}} E^{m-1} \xleftarrow{\pi_{m-1}} E^m = E$$

with irreducible nonsingular center in E^i and exceptional variety $C_{i+1} \subset E^{i+1}$ for $i = 0, \ldots, m-1$. The variety E^0 is created at some stage of the global resolution process as the exceptional variety of a blowing-up with center D and is isomorphic to a projective space bundle $\Pi : E^0 \to D$ over D.

There are two kinds of intersections of E with other components of $h^{-1}(f^{-1}\{0\})$. We have the repeated strict transforms $C_1^{(m)},\dots,C_m^{(m)}$ in E of the exceptional varieties $C_1,\dots,C_m$; and furthermore we have the repeated strict transforms $C_i^{(m)}$ in E of varieties C_i, with $i\in T$, (of codimension one) in E^0.

For each $i\in T\cup\{1,\dots,m\}$ the strict transform $C_i^{(m)}$ of C_i in E is (an irreducible component of) the intersection of E with exactly one other component of $h^{-1}(f^{-1}\{0\})$. Let this component have numerical data (N_i,ν_i) and set $\alpha_i=\nu_i-(\nu/N)N_i$. (The numbers α_i, $i\in T\cup\{1,\dots,m\}$, occur in the expression for the residue of the candidate-pole $-\nu/N$ for $Z(s)$, see THEOREM 1.2.)

There are *basic relations* (B1 and B2) between the α_i, $i\in T$, and there is for each $i\in\{0,\dots,m-1\}$ an *additional relation* (A) expressing α_{i+1} in terms of the α_k for $k\in T\cup\{1,\dots,i\}$.

For the applications on the poles of $Z(s)$, we choose the number field K «large enough», meaning that the resolution (X,h) over $\mathbb{Q}^a$ is entirely defined over K itself.

We now suppose that the fixed exceptional variety E satisfies $\chi(\overset{\circ}{E})=0$ and that there is no E_i, with $i\in S\setminus\{j\}$, intersecting E with $\nu_i/N_i=\nu/N$. Denote by $\mathcal{R}$ the contribution of E to the residue of the candidate-pole $-\nu/N$ for $Z(s)$.

Surfaces. — When $n=2$, the surface E^0 is created by blowing-up a point or a nonsingular curve D. In the first case $E^0\cong\mathbb{P}^2$ and in the latter E^0 is a ruled surface $\Pi: E^0\to D$ over D.

By the formula for $Z(s)$ of THEOREM 1.2 we can express $\mathcal{R}$ in this case as follows. Set $\overset{\circ}{C}_i= C_i^{(m)}\setminus\bigcup_{\ell\neq i}C_\ell^{(m)}$ and $\alpha_i=\nu_i-(\nu/N)N_i$ for $i\in T\cup\{1,\dots,m\}$. Then

$$\begin{aligned}\mathcal{R}=\operatorname{card}\overset{\circ}{E}&\\ &+(q-1)\sum_{i\in T\cup\{1,\dots,m\}}\frac{\operatorname{card}\overset{\circ}{C}_i}{q^{\alpha_i}-1}\\ &\qquad+(q-1)^2\sum_{\substack{\{i,j\}\subset T\cup\{1,\dots,m\}\\ i\neq j}}\frac{\operatorname{card}(C_i^{(m)}\cap C_j^{(m)})}{(q^{\alpha_i}-1)(q^{\alpha_j}-1)},\end{aligned}$$

where $\operatorname{card}\overset{\circ}{E}$ and $\operatorname{card}\overset{\circ}{C}_i$ are the number of $\bar{K}$-rational points of the reduction of respectively $\overset{\circ}{E}$ and $\overset{\circ}{C}_i$ modulo $\mathfrak{p}R_{\mathfrak{p}}$.