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ON MICROLOCAL b-FUNCTION

BY

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RÉSUMÉ. — Soit f un germe de fonction holomorphe en n variables. En utilisant des opérateurs différentiels microlocals, on introduit la notion de b-fonction microlocale $\tilde{b}_f(s)$ de f, et on démontre que $(s+1)\tilde{b}_f(s)$ coïncide avec la b-fonction (i.e. le polynôme de Bernstein) de f. Soient R_f les racines de $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$ et $m_\alpha(f)$ la multiplicité de $\alpha \in R_f$. On démontre $R_f \subset [\alpha_f, n-\alpha_f]$ et $m_\alpha(f) \le n - \alpha_f - \alpha + 1$ ($\le n - 2\alpha_f + 1$). Le théorème de type Thom-Sebastiani pour b-fonction est aussi démontré sous une hypothèse raisonnable.

ABSTRACT. — Let f be a germ of holomorphic function of n variables. Using microlocal differential operators, we introduce the notion of microlocal b-function $\tilde{b}_f(s)$ of f, and show that $(s+1)\tilde{b}_f(s)$ coincides with the b-function (i.e. Bernstein polynomial) of f. Let R_f be the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, and $m_\alpha(f)$ the multiplicity of $\alpha \in R_f$. Then we prove $R_f \subset [\alpha_f, n-\alpha_f]$ and $m_\alpha(f) \le n - \alpha_f - \alpha + 1$ ($\le n - 2\alpha_f + 1$). The Thom-Sebastiani type theorem for b-function is also proved under a reasonable hypothesis.

Introduction

Let f be a holomorphic function defined on a germ of complex manifold (X, x). The b-function (i.e., Bernstein polynomial) $b_f(s)$ of f is defined by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.1) \qquad b(s)f^s = Pf^{s+1} \quad \text{in } \mathcal{O}_{X,x}[f^{-1}][s]f^s$$

for $P \in \mathcal{D}_{X,x}[s]$. Let $\delta(t-f)$ denote the delta function on $X' := X \times \mathbb{C}$ with support $\{f = t\}$, where t is the coordinate of $\mathbb{C}$. Then, setting $s = -\partial_t t$, f^s and $\delta(t-f)$ satisfy the same relation (see for example [8]). So f^s in (0.1) can be replaced by $\delta(t-f)$, and f^{s+1} by $t\delta(t-f)$. We define the

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microlocal b-function $\tilde{b}_f(s)$ by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.2) \qquad b(s)\delta(t-f) = P\partial_t^{-1}\delta(t-f) \quad \text{in } \mathcal{O}_{X,x}[\partial_t, \partial_t^{-1}]\delta(t-f)$$

for $P \in \mathcal{D}_{X,x}[\partial_t^{-1}, s]$. Here we can also allow for P a microdifferential operator [4], [6], [17] satisfying a condition on the degree of t and ∂_t (see (1.4)). We have :

PROPOSITION 0.3. — $b_f(s) = (s+1)\tilde{b}_f(s)$.

See (1.5). The microlocal b-function $\tilde{b}_f(s)$ is sometimes easier to treat than the b-function $b_f(s)$. Let R_f be the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, $m_\alpha(f)$ the multiplicity of $\alpha \in R_f$, and $n = \dim X$. Then, using the duality of filtered $\mathcal{D}$-Modules [15] and the theory of Hodge Modules [12], we prove

THEOREM 0.4. — $R_f \subset [\alpha_f, n - \alpha_f]$.

THEOREM 0.5. — $m_\alpha(f) \leq n - \alpha_f - \alpha + 1 \quad (\leq n - 2\alpha_f + 1)$.

See (2.8), (2.10).

The estimate (0.4) is optimal because $\max R_f = n - \alpha_f$ in the quasi-homogeneous isolated singularity case. See also remark after (2.8) below. Note that $R_f \subset \mathbb{Q}$ and $\alpha_f > 0$ by [4], and (0.5) is an improvement of $m_\alpha(f) \leq n - \delta_{\alpha,1}$ (with $\delta_{\alpha,1}$ Kronecker's delta) which is shown in [9] as a corollary of the relation with Deligne's vanishing cycle sheaf $\varphi_f\mathbb{C}_X$ [2] (see also [5]). This relation implies for example that $\exp(2\pi i\alpha)$ for $\alpha \in R_f$ are the eigenvalues of the monodromy on $\varphi_f\mathbb{C}_X$. But $\varphi_f\mathbb{C}_X$ cannot be replaced with the reduced cohomology of a Milnor fiber at x as in the isolated singularity case, because we have to take the Milnor fibration at several points of Sing $f^{-1}(0)$ even when we consider the b-function of f at x. See (2.12) below.

Let T_u and T_s denote respectively the unipotent and semisimple part of the monodromy T on $\varphi_f\mathbb{C}_X$. Let $\varphi_f^\alpha\mathbb{C}_X = \mathrm{Ker}(T_s - \exp(-2\pi i\alpha))$ (as a shifted perverse sheaf), and $N = \log T_u/2\pi i$. In the proof of (0.5), we get also :

PROPOSITION 0.6. — *We have* $N^{r+1} = 0$ *on* $\varphi_f^\alpha\mathbb{C}_X$ *for* $\alpha \in [\alpha_f, \alpha_f + 1)$ *and* $r = [n - \alpha_f - \alpha]$. *In particular,* $N^{r+1} = 0$ *on* $\varphi_f\mathbb{C}_X$ *for* $r = [n - 2\alpha_f]$.

For the proof of (0.4)–(0.6), we use the filtration V (similar to that in [5], [9]) defined on the $\mathcal{D}_{X,x}[t, \partial_t, \partial_t^{-1}]$-module $\widetilde{\mathcal{B}}_f$ generated by the delta function $\delta(t-f)$. Note that (0.3) may be viewed as an extension of Malgrange's result [8] to the nonisolated singularity case (see (1.7) below), and in the isolated singularity case, (0.4)–(0.6) can be deduced from results of [8], [19], [20] (and [18]) using an argument as in [14]. In the nondegenerate Newton boundary case [7], we get an estimate of α_f using the Newton polyhedron (see (3.3)). The idea of its proof is essentially same as [16].

Let g be a holomorphic function on a germ of complex manifold (Y, y). Let $Z = X \times Y, z = (x, y)$, and $h = f + g \in \mathcal{O}_{Z,z}$. We define R_g, R_h as above. Then we have :

PROPOSITION 0.7. — $R_f + R_g \subset R_h + \mathbb{Z}_{\leq 0}$, $R_h \subset R_f + R_g + \mathbb{Z}_{\geq 0}$.

THEOREM 0.8. — *Assume there is a holomorphic vector field ξ such that $\xi g = g$. Then we have $R_f + R_g = R_h$, and*

$$m_\gamma(h) = \max_{\alpha+\beta=\gamma} \{m_\alpha(f) + m_\beta(g) - 1\}.$$

See (4.3)–(4.4). Here $\mathbb{Z}_{\geq 0}$ (or $\mathbb{Z}_{\leq 0}$) is the set of nonnegative (or nonpositive) integers. In the case where f and g have isolated singularities, (0.7)–(0.8) can be easily deduced from results of MALGRANGE [8], [10] (see (4.6) below), and (0.8) was first obtained by [21] in this case. Note that (0.8) is not true in general if the hypothesis is not satisfied. See (4.8) below.

1. Microlocal b-function

1.1. — Let X be a complex manifold of pure dimension n, and $x \in X$. Let $\mathcal{O} = \mathcal{O}_{X,x}, \mathcal{D} = \mathcal{D}_{X,x}$. We define rings $\mathcal{R}, \widetilde{\mathcal{R}}$ by

$$\mathcal{R} = \mathcal{D}[t, \partial_t], \quad \widetilde{\mathcal{R}} = \mathcal{D}[t, \partial_t, \partial_t^{-1}], \tag{1.1.1}$$

where t, ∂_t satisfy the relation $\partial_t t - t\partial_t = 1$, and $\mathcal{D}[t, \partial_t] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[t, \partial_t]$, etc. We define the filtration V on $\mathcal{R}, \widetilde{\mathcal{R}}$ by the differences of the degrees of t and ∂_t :

$$V^p\mathcal{R} = \sum_{i-j \geq p} \mathcal{D}t^i\partial_t^j \quad (\text{same for } \widetilde{\mathcal{R}}). \tag{1.1.2}$$

Then we have :

$$\begin{cases} V^p\mathcal{R} = t^pV^0\mathcal{R} = V^0\mathcal{R}t^p \quad (p > 0), \\ V^{-p}\mathcal{R} = \sum_{0 \leq j \leq p} \partial_t^jV^0\mathcal{R} = \sum_{0 \leq j \leq p} V^0\mathcal{R}\partial_t^j \quad (p > 0), \\ V^p\widetilde{\mathcal{R}} = \partial_t^{-p}V^0\widetilde{\mathcal{R}} = V^0\widetilde{\mathcal{R}}\partial_t^{-p}. \end{cases} \tag{1.1.3}$$

1.2. — Let $f \in \mathcal{O}$ such that $f(0) = 0$ and $f \neq 0$. Let

$$(1.2.1) \qquad \mathcal{B}_f = \mathcal{O}[\partial_t]\delta(t-f), \quad \widetilde{\mathcal{B}}_f = \mathcal{O}[\partial_t, \partial_t^{-1}]\delta(t-f),$$

where $\mathcal{O}[\partial_t]\delta(t-f)$ is a free module of rank one over $\mathcal{O}[\partial_t]$ $(= \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])$ with a basis $\delta(t-f)$ (similarly for $\widetilde{\mathcal{B}}_f$). Here $\delta(t-f)$ denotes the delta function supported on $\{f = t\}$ (see remark below). We have a structure of $\mathcal{R}$-module and $\widetilde{\mathcal{R}}$-module on $\mathcal{B}_f$ and $\widetilde{\mathcal{B}}_f$ respectively by

$$(1.2.2) \qquad \begin{cases} \xi\big(a\partial_t^i\delta(t-f)\big) = (\xi a)\partial_t^i\delta(t-f) - (\xi f)a\partial_t^{i+1}\delta(t-f), \\ t\big(a\partial_t^i\delta(t-f)\big) = fa\partial_t^i\delta(t-f) - ia\partial_t^{i-1}\delta(t-f) \end{cases}$$

for $a \in \mathcal{O}$ and $\xi \in \Theta_{X,x}$. We define a decreasing filtration G on $\mathcal{B}_f$, $\widetilde{\mathcal{B}}_f$ by

$$(1.2.3) \qquad G^p\mathcal{B}_f = V^p\mathcal{R}\delta(t-f), \quad G^p\widetilde{\mathcal{B}}_f = V^p\widetilde{\mathcal{R}}\delta(t-f),$$

and an increasing filtration F by

$$(1.2.4) \qquad F_p\mathcal{B}_f = \bigoplus_{0 \le i \le p} \mathcal{O}\partial_t^i\delta(t-f), \quad F_p\widetilde{\mathcal{B}}_f = \bigoplus_{i \le p} \mathcal{O}\partial_t^i\delta(t-f)$$

Then we have :

$$(1.2.5) \qquad \partial_t^i : G^p\widetilde{\mathcal{B}}_f \xrightarrow{\sim} G^{p-i}\widetilde{\mathcal{B}}_f, \quad \partial_t^i : F_p\widetilde{\mathcal{B}}_f \xrightarrow{\sim} F_{p+i}\widetilde{\mathcal{B}}_f,$$

$$(1.2.6) \qquad \mathcal{D}_{X,x}[s](F_p\widetilde{\mathcal{B}}_f) \subset G^{-p}\widetilde{\mathcal{B}}_f.$$

Remark. — The $\mathcal{R}$-module $\mathcal{B}_f$ is identified with the germ at $(x,0)$ of the direct image of $\mathcal{O}_X$ as $\mathcal{D}$-Module by the closed embedding i_f defined by the graph of f, where t is identified with the coordinate of $\mathbb{C}$. See [4] and [17].

1.3 Definition. — The *b-function* $b_f(s)$ (resp. *microlocal b-function* $\tilde{b}_f(s)$) is defined by the minimal polynomial of the action of $s := -\partial_t t$ on $\mathrm{Gr}_G^0\mathcal{B}_f$ (resp. $\mathrm{Gr}_G^0\widetilde{\mathcal{B}}_f$).

REMARK. — Since $\mathrm{Gr}_V^0\mathcal{R} = \mathrm{Gr}_V^0\widetilde{\mathcal{R}} = \mathcal{D}[s]$, $b_f(s)$ (resp. $\tilde{b}_f(s)$) is the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(1.3.1) \qquad b(s)\delta(t-f) = P\delta(t-f)$$