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LIOUVILLE THEOREMS BASED ON SYMMETRIC DIFFUSIONS

PAR

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Dedicated to Professor M. Fukushima on his 60th birthday

RÉSUMÉ. — On étudie dans cet article le théorème de Liouville pour les fonctions sousharmoniques en se basant sur l'espace de Dirichlet. L'intégrale de Hellinger permet d'écrire les théorèmes de Liouville pour les fonctions sousharmoniques sans hypothèse de régularité sur une fonction exhaustive donnée et d'examiner avec précision la croissance des fonctions sousharmoniques non constantes.

ABSTRACT. — In this article, we study Liouville properties for subharmonic functions based on the Dirichlet space theory. In order to describe Liouville theorems for subharmonic functions, we will use the Hellinger integral which eliminates the smoothness requirement on the given exhaustion function and enables us to examine the increasing order of non-constant subharmonic functions.

1. Introduction

The study on Liouville theorems has been developed in geometry as well as in complex analysis. Geometrical Liouville theorems ordinarily state that if a Riemannian manifold enjoys certain condition on the curvature, then it does not admit non-constant bounded subharmonic function (*e.g.* [C-Y]). In complex analysis, Liouville theorems assert that any bounded plurisubharmonic function or occasionally pluriharmonic function does not exist except constant, whenever the underlying complex manifold is parabolic in a certain sense (*e.g.* [K], [T1] and [T2]). N. Sibony and P.M. Wong [S-W] suggested that this sort of assertion is usually related to the vanishing of the capacity of the infinity. The notion of capacity for the value distribution theory was originally dealt with by W. Stoll (*e.g.* [Stol]).

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In the mean time, the capacity is expressed by the Dirichlet integral of 0-equilibrium potential with respect to the corresponding self-adjoint differential operator. M. Fukushima pointed out in [F2] that the vanishing of the Dirichlet integral concerns the non-transience of the corresponding Hunt process. In the recent development of Dirichlet space theory, H. Ôkura [Ok3] found a recurrence criterion for Hunt process in terms of the Hellinger integral which is attributed to [H]. The method was utilized to reproduce a sharp capacitary estimate by K.T. Sturm [Stu2]. Their arguments are based upon the stochastic metric introduced by M. Biroli and U. Mosco in [B-M2].

Throughout this paper, we focus our attention on the case that the Dirichlet space has the strong local property, that is, the case that the corresponding Hunt process is a diffusion with no killing inside. We start with an inequality which will be the primary tool for our Liouville theorems formulated in Section 3. Our theorems not only handle the family of subharmonic functions locally in the domain of the Dirichlet space but imply some of the known results on the absence of bounded functions with subharmonicity such as [C-Y], [T1] and [Stu1]. Our method also provides a Liouville theorem without exhaustion function, which is directly applicable to the plurisubharmonic functions without the approximating procedures as in [T1].

As for notions and notations, the author recommends the reader to consult the book [F-O-T]. The author expresses his thanks to Professor M. Fukushima and Professor M. Takeda for their heartfelt encouragement and the proof of Lemma 1 which is shorter than the author's original one.

2. Green's formula

We denote a regular Dirichlet space on $L^2(X,m)$ with the strong local property by $(\mathcal{E},\mathcal{F})$, where X is a locally compact Hausdorff space and m is a Radon measure with $\operatorname{supp}[m]=X$. In this section, we deal with the case that the underlying space X has a continuous exhaustion function $\rho\in\mathcal{F}_{\mathrm{loc}}$. Throughout this paper, all elements in $\mathcal{F}_{\mathrm{loc}}$ are assumed to be quasi-continuous already.

We use the following notations:

$$\begin{aligned} B(s) &= \{x\in X\ ;\ \rho(x)<s\}, && \text{for}\quad s>0\,,\\ B(r,s) &= \{x\in X\ ;\ r<\rho(x)<s\}, && \text{for}\quad s>r>0\,,\\ e_{\langle u,v\rangle}(s) &= \mu_{\langle u,v\rangle}\big(B(s)\big), && \text{for}\quad u,v\in\mathcal{F}_{\mathrm{loc}}\,,\end{aligned}$$

where $\mu_{\langle u,v\rangle}$ is the co-energy of $u,v\in\mathcal{F}_{\mathrm{loc}}$.

The measure $\mu_{\langle u,u\rangle}$ (resp. $e_{\langle u,u\rangle}$) is denoted by $\mu_{\langle u\rangle}$ (resp. $e_{\langle u\rangle}$).

MAIN LEMMA (Green's formula). — *If $u \in \mathcal{F}_{\mathrm{loc}}$ and $v \in \mathcal{F}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}$, then*

$$\int_r^R e_{\langle u,v\rangle}(s)\,\mathrm{d}s - \int_X \mathrm{d}\mu_{\langle u,w_{r,R}v\rangle} = \int_{B(r,R)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle}$$

holds for $w_{r,R} = R - (\rho \wedge R) \vee r$, where

$$x \vee a = \max\{x,a\} \quad \textit{and} \quad x \wedge b = \min\{x,b\}.$$

Especially, if u is $\mathcal{E}$-subharmonic and v is non-negative, then

$$\int_r^R e_{\langle u,v\rangle}(s)\,\mathrm{d}s \le \int_{B(r,R)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle}\,.$$

Proof. — Since the assertion is local, it suffices to consider the case that $u \in \mathcal{F}$ and $v \in \mathcal{F}_b$. By using Theorem 3.2.2 in [F-O-T], we can derive the desired identity as follows:

$$\begin{aligned}\int_r^R \mathrm{e}_{\langle u,v\rangle}(s)\,\mathrm{d}s &= \int_r^R \int_{B(s)} \mathrm{d}\mu_{\langle u,v\rangle}\,\mathrm{d}s = \int_X w_{r,R}\,\mathrm{d}\mu_{\langle u,v\rangle}\\ &= \int_X \mathrm{d}\mu_{\langle u,w_{r,R}v\rangle} - \int_X v\,\mathrm{d}\mu_{\langle u,w_{r,R}\rangle}\\ &= \int_X \mathrm{d}\mu_{\langle u,w_{r,R}v\rangle} + \int_{B(r,R)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle},\end{aligned}$$

where the last equality follows from the identity in [M] (see also [Ok3, Lemma 2.1 (i)]). The first term in the last expression equals $2\mathcal{E}(w_{r,R}v,u)$, which is non-positive when u is subharmonic and v is non-negative. □

REMARK 1. — We consider the case that

$$\mathcal{F} = H^1(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N)\,;\ \frac{\partial u}{\partial x^i} \in L^2(\mathbb{R}^N),\ i = 1,\dots,N\Big\},$$

$$\mathcal{E}(u,v) = \tfrac12 \int_{\mathbb{R}^N} (\nabla u, \nabla v)\,\mathrm{d}V \quad \text{for } u,v \in \mathcal{F},$$

where V is the Lebesgue measure. If a bounded domain D is exhausted by a smooth function ρ as $D = \{\rho < r\} \subset \mathbb{R}^N$ and $\mathrm{d}\rho \neq 0$ is satisfied on ∂D, we can verify the following convergences for functions u, v in C^2:

$$\begin{aligned}&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{r-\varepsilon}^{r} e_{\langle u,v\rangle}(s)\,\mathrm{d}s = \int_D (\nabla u, \nabla v)\,\mathrm{d}V,\\ &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_X \mathrm{d}\mu_{\langle u,w_{r-\varepsilon,r}v\rangle} = -\int_D v\,\triangle u\,\mathrm{d}V,\\ &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B(r-\varepsilon,r)} v\,\mathrm{d}\mu_{\langle u,\rho\rangle} = \int_{\partial D} v\,\frac{\partial u}{\partial \nu}\,\mathrm{d}S,\end{aligned}$$

where $\partial u/\partial \nu$ denotes the outer normal derivative of u and S stands for the volume element on ∂D. Therefore, our formula implies the classical Green's formula.

In the sequel, we also use the following lemma.

LEMMA 1. — *A function u in $\mathcal{F}_{\mathrm{loc}}$ is constant quasi-everywhere on X if $\mu_{\langle u\rangle}$ vanishes.*

Proof. — By the strong local property, $\mu_{\langle (u-\alpha)\vee 0\rangle} = I_{\{u>\alpha\}}\mu_{\langle u\rangle}$ for any $\alpha \in \mathbb{R}$ and accordingly $\mu_{\langle (u-\alpha)\vee 0\rangle}$ vanishes under the stated condition. This implies the $\mathcal{E}$-harmonicity of $(u-\alpha)\vee 0$ and consequently

$$p_t\big[(u-\alpha)\vee 0\big] = (u-\alpha)\vee 0 \qquad (\forall\,\alpha\in\mathbb{R},\ \forall\,t>0)$$

for the semi-group $\{p_t\}$ generated by $(\mathcal{E},\mathcal{F})$. The p_t-invariance of $\{u\le\alpha\}$ follows from this identity. From the irreducibility of the diffusion it turns out that u must equal $\inf\{\alpha\,;\ \mathrm{Cap}(X\setminus\{u\le\alpha\})=0\}$ quasi-everywhere on X. $\square$

3. The Liouville Theorems

We start with the case that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet space on $L^2(X,m)$ with the strong local property which generates an irreducible diffusion and that X has a continuous exhaustion function $\rho\in\mathcal{F}_{\mathrm{loc}}$. For $u\in\mathcal{F}_{\mathrm{loc}}$, we set

$$m(u,s) = \operatorname{ess\,sup}\big\{u(x)\,;\ \rho(x)<s\big\},$$

which is equal to $\inf\{\alpha\,;\ \mathrm{Cap}(B(s)\setminus\{u\le\alpha\})=0\}$ by the quasi-continuity of u. We further set

$$h^*(s) = h(s) - \sum_{\xi\le s}\big(h(\xi+)-h(\xi)\big)$$

for any left-continuous increasing function h.

We shall particularly need $e^*_{\langle u\rangle}(\cdot)$.

LEMMA 2. — *If $u\in\mathcal{F}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}$ is non-negative and $\mathcal{E}$-subharmonic, then*

$$\text{(i)}\quad \int_r^R e_{\langle u\rangle}(s)\,\mathrm{d}s \le m(u,R)\sqrt{e_{\langle\rho\rangle}(R)-e_{\langle\rho\rangle}(r)} \times\sqrt{e^*_{\langle u\rangle}(R)-e^*_{\langle u\rangle}(r)}\,,$$

$$\text{(ii)}\quad \int_r^R e^*_{\langle u\rangle}(s)\,\mathrm{d}s \le m(u,R)\sqrt{e_{\langle\rho\rangle}(R)-e_{\langle\rho\rangle}(r)} \times\sqrt{e^*_{\langle u\rangle}(R)-e^*_{\langle u\rangle}(r)}\,.$$