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THE THEORY OF DIFFERENTIAL INVARIANTS AND KDV HAMILTONIAN EVOLUTIONS

BY GLORIA MARÍ BEFFA (*)

ABSTRACT. — In this paper I prove that the second KdV Hamiltonian evolution associated to $\mathrm{SL}(n,\mathbb{R})$ can be view as the most general evolution of projective curves, invariant under the $\mathrm{SL}(n,\mathbb{R})$-projective action on $\mathbb{RP}^{n-1}$, provided that certain integrability conditions are satisfied. This way, I establish a very close relationship between the theory of geometrical invariance, and KdV Hamiltonian evolutions. This relationship was conjectured in [4].

RÉSUMÉ. — LA THÉORIE DES INVARIANTS DIFFÉRENTIELS ET LES ÉVOLUTIONS HAMILTONIENNES DE KDV. — Dans cet article, je prouve que la seconde évolution hamiltonienne de KdV, associée au groupe $\mathrm{SL}(n,\mathbb{R})$, peut être considérée comme l'évolution la plus générale des courbes projectives qui sont invariantes par l'action projective de $\mathrm{SL}(n,\mathbb{R})$ sur $\mathbb{RP}^{n-1}$, si une certaine condition d'integrabilité est satisfaite. Je mets alors en évidence une connection très étroite entre la théorie d'invariance géométrique et les évolutions hamiltoniennes de KdV. Cette relation a été conjecturée en [4].

1. Introduction

Consider the following problem: Let $\phi(t,\theta) \in \mathbb{RP}^{n-1}$ be a family of projective curves. We ask the following question: is there a formula describing the most general evolution for ϕ of the form

$$\phi_t = F(\phi, \phi', \phi'', \ldots)$$

invariant under the projective action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{RP}^{n-1}$? Here

$$\phi' = \phi_\theta = \frac{\mathrm{d}\phi}{\mathrm{d}\theta}, \quad \phi_t = \frac{\mathrm{d}\phi}{\mathrm{d}t}\cdot$$

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The projective action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{RP}^{n-1}$ is the one induced on $\mathbb{RP}^{n-1}$ by the usual action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ via the lift

$$\mathbb{RP}^{n-1} \longrightarrow \mathbb{R}^n, \quad \phi \longmapsto (1,\phi).$$

As we showed in [4], such a formula can be found using the theory of projective differential invariance. In fact, one can prove that any evolution of projectives curves which is invariant under $\mathrm{SL}(n,\mathbb{R})$ can always be written as

$$\phi_t = \mu \mathcal{I} \tag{1.1}$$

where $\mathcal{I}$ is a *vector of differential invariants* for the action and μ is a particular (fixed) matrix of *relative invariants*, whose explicit formula was found in [4]. Roughly speaking, if a group G acts on a manifold M, one can define an action of the group on a given jet bundle $J^{(k)}$ of order k, where $J^{(k)}$ is the set of equivalence classes of submanifolds modulo k-order contact. This action, in coordinates looks like

$$\begin{aligned} G \times J^{(k)} &\longrightarrow J^{(k)}, \\ (g, u_K) &\longmapsto (gu)_K, \end{aligned}$$

for any differential subindex K of order less or equal to k, and it is called *the prolonged action.* A differential invariant is a map

$$I \colon J^{(k)} \longrightarrow \mathbb{R}$$

which is invariant under the prolonged action. A relative differential invariant is a map

$$J \colon J^{(k)} \longrightarrow \mathbb{R}$$

whose value gets multiplied by a factor under the prolonged action. The factor is usually called the multiplier. In our particular case, their infinitesimal definitions are given in the second part of section 2. Differential invariants and relative invariants are the tools one uses to describe invariant evolutions.

These two concepts belong to the theory of Klein geometries and geometric invariants which had its high point last century before the appearance of Cartan's approach to differential geometry. It is also closely related to equivalence problems. Namely, one poses the question of equivalence of two geometrical objects under the action of a certain group, that is, when can one of those objects be taken to the other one using a transformation belonging to the given group? For example, given two curves on the plane, when are they equivalent under an Euclidean motion? or, when are they the same curve, up to parametrization? *etc.* One answer can be given in terms of *invariants*, that is, expressions depending on the objects under study and that do not change under the action of the group. If two objects are to be equivalent, they must have the same invariants. If these invariants are functions on some jet space (for example, if they depend on the curve and its derivatives with respect to the parameter), then they are called differential invariants. In the case of curves on the Euclidean plane under the action of the Euclidean group, the basic differential invariant is known to be the Euclidean curvature, and any other differential invariant will be a function of the curvature and its derivatives. In the case of immersions

$$\phi\colon \mathbb{R} \longrightarrow \mathbb{RP}^1,$$

with $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathbb{RP}^1$, the basic differential invariant is classically known to be the *Schwarzian derivative* of ϕ,

$$S(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\Big(\frac{\phi''}{\phi'}\Big)^2.$$

Within the natural scope of the study of equivalence problems and their invariants lies also the description of invariant differential equations, symmetries, relative invariants, *etc.* For example, recently Olver *et al.* [12] used these ideas to characterize all scalar evolution equations invariant under the action of a subgroup of the projective group in the plane, a problem of interest in the theory of image processing. See Olver's book [11] for an account of the state of the subject.

A subject apparently unrelated to the Theory of differential invariance is the subject of Hamiltonian structures of partial differential equations, integrability and, in general, of infinite dimensional Poisson structures. The so-called KdV Poisson brackets lie within this area. These brackets were defined by Adler [1] in an attempt to generalize the bi-Hamiltonian character of the Korteweg-deVries (KdV) equation and its integrability. He defined a family of second Hamiltonian structures with respect to which the generalized higher-dimensional KdV equations could also be written as Hamiltonian systems. Jacobi's identity for these brackets was proved by Gel'fand and Dikii in [3]. These Poisson structures are called *second Hamiltonian KdV structures* or *Adler-Gel'fand-Dikii brackets*, and they are defined on the manifold of smooth Lax operators. Since the original definition of Adler was quite complicated and not very intuitive, alternative definitions have been subsequently offered by several authors, most notably by Kupershmidt and Wilson in [7], and by Drinfel'd and Sokolov in [2]. Once the second Hamiltonian structure was found, the integrability of generalized KdV equations was established via the usual construction of a sequence of Hamiltonian structures with commuting Hamiltonian operators. In this paper I will restrict to the case of the $\mathrm{SL}(n,\mathbb{R})$ Adler-Gel'fand-Dikii bracket, although brackets have been given for other groups (Drinfel'd and Sokolov described their definition for any semisimple Lie algebra). The second Hamiltonian Structure in this hierarchy of KdV brackets coincides with the usual second Poisson bracket for the KdV equation, that is, the canonical Lie-Poisson bracket on the dual of the Virasoro algebra. This is the only instance in which the second KdV bracket is linear.

The relationship between Lax operators (scalar n-th order ODE's) and projective curves was established by the classics and clearly described by Wilczynski in [13]. More recently (see [12]) the topology of these curves was used to identify one of the invariants of the symplectic leaves of the Adler–Gel'fand–Dikii Poisson foliation. Some comments with respect to the role of projective curves in these brackets can be found in [14] and [7]. In [4] it was conjectured that the second KdV Hamiltonian evolution and the general evolution for projective curves (1.1) found in [4] were, essentially, the same evolution under a 1-to-1 (up to $\mathrm{SL}(n,\mathbb{R})$ action) correspondence between Lax operators and projective curves. The only condition that needed to be imposed was that certain invariant combination of the components of the invariant vector $\mathcal{I}$ in (1.1) should be integrable to define the gradient of certain Hamiltonian operator (one can even describe the evolution so that both I and Hamiltonian coincide after the identification).

In this paper I prove this conjecture. Namely, I prove that there exists an invariant matrix $\mathcal{M}$, invertible, such that, if H is the *pseudo-differential operator associated to an operator* $\mathcal{H}$, and if

$$H = \mathcal{M}\mathcal{I}$$

then, whenever ϕ evolves following (1.1) with general invariant vector $\mathcal{I}$, then their associated Lax operators (associated in the sense of [4] and described again in the next chapter) will evolve following an AGD-evolution with Hamiltonian operator $\mathcal{H}$. I also prove the conjectured shape of $\mathcal{M}$, namely, lower triangular along the transverse diagonal with ones down the transverse diagonal and zeroes on the diagonal inmediately below the transverse one. The proof is based on a manipulation of Wilson's antiplectic pair for the $\mathrm{GL}(n,\mathbb{R})$-AGD bracket and on the comparison of the resulting formulas with the invariant formulas (1.1).