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COMPARING HEAT OPERATORS THROUGH LOCAL ISOMETRIES OR FIBRATIONS

BY MANLIO BORDONI (*)

ABSTRACT. — Our aim is to generalize and improve the Kato's inequality, which compares the trace of the heat kernel of a compact Riemannian manifold with the one of a finite-sheeted covering of it. A comparison with the heat kernel of a suitable space-form gives, as a consequence, an analogous of Kato's inequality for non compact manifolds, which improves the classical inequality when the manifolds are compact. We get another generalization for local isometries, which are no more supposed to be covering maps (as a typical example, we apply this to the exponential map). Last, we consider Riemannian submersions with minimal fibers.

RÉSUMÉ. — COMPARAISON ENTRE OPÉRATEURS DE LA CHALEUR PAR ISOMÉTRIES LOCALES OU FIBRATIONS. — Notre but est de généraliser et d'améliorer l'inégalité de Kato, qui compare la trace du noyau de la chaleur d'une variété riemannienne compacte donnée à celle d'un revêtement riemannien fini de la variété. Une comparaison avec le noyau de la chaleur d'une variété simplement connexe de courbure constante convenablement choisie donne, comme conséquence, un analogue de l'inégalité de Kato qui améliore l'inégalité classique quand le revêtement n'est pas compact. On obtient une généralisation dans le cas où les variétés sont reliées par une isométrie locale (qui n'est pas obligatoirement un revêtement, un exemple typique étant donné par l'application exponentielle). Enfin, on traite le cas des submersions riemanniennes à fibres minimales.

1. Introduction

Let (M,g) be any connected Riemannian manifold of finite dimension n. Let us denote by $\Delta_M = \Delta_{(M,g)}$ the Laplace-Beltrami operator acting on functions and let us consider the heat equation:

$$\Big(\Delta_M + \frac{\partial}{\partial t}\Big)u = O, \tag{1.1}$$

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with Dirichlet or Neumann condition on the boundary if M has a nonempty boundary ∂M. The corresponding heat kernel will be denoted by $p_M(t,x,y)$ when the boundary is empty, $p^D_M(t,x,y)$ or $p^N_M(t,x,y)$ resp. when the boundary condition is Dirichlet's or Neumann's one. For a non compact manifold, we shall consider p_M to be the unique minimal heat kernel, *i.e.* the limit of the Dirichlet heat kernels of regular compact domains exhausting M; if M is complete and if its Ricci curvature is bounded from below, then p_M is the unique heat kernel on M (see [9, p. 189]). If M is compact, the spectrum of Δ_M is a discrete sequence $\{\lambda_i(M)\}_{i=0,1,2,\dots}$ (each eigenvalue is repeated according to its finite multiplicity); in this case, we shall also consider the trace $Z_M(t)$ of the heat operator $\mathrm{e}^{-\Delta_M t}$ (with positive t):

$$Z_M(t) = \sum_{i=0}^{+\infty} \mathrm{e}^{-\lambda_i(M)t}$$

and similar expressions for $Z^D_M(t)$ and $Z^N_M(t)$.

Given a mapping $f\colon (M',g') \to (M,g)$, our aim is to compare the heat kernels of the two manifolds, under suitable assumptions for f. It turns out that a good assumption is that f satisfies the following *Fubini's property* for every continuous function u on M':

$$(1.2)\qquad \int_{M'} u(x')\,\mathrm{d}v_{g'}(x') = \int_M \Big\{ \int_{f^{-1}(x)} u_{|f^{-1}(x)}(y)\,\mathrm{d}v_{g'_x}(y) \Big\}\,\mathrm{d}v_g(x)$$

where $v_{g'}, v_g$ are the measures canonically associated to the metrics g', g, and where $v_{g'_x}$ denotes the measure associated to the metric g'_x induced on $F_x = f^{-1}(x)$ by g'.

Notice that, by Sard's theorem, if f is smooth on the outside U' of a closed subset of measure zero in M', and if $\dim M' > \dim M$, then the intersection of F_x with U' is a submanifold for almost every x, so that the integrals which occur in the formula (1.2) make sense. By the coarea formula (see [8, thm. 13.4.2]), this may be extended to the case where f is only a Lipschitz map. In this case, the differential $\mathrm{d}f_{x'}$ exists for a.e. x' and, considering its restriction to the orthogonal complement $H_{x'}$ of $T_{x'}(F_{f(x')})$ in $T_{x'}M'$, we may define its Jacobian as the determinant of this restriction. By Corollary 13.4.6 of [8], condition (1.2) is, in this case, equivalent to saying that this Jacobian is a.e. equal to ± 1. The property (1.2) is automatically satisfied for instance by Riemannian submersions and coverings, or by local isometries.

If f is a fibration of compact manifolds with typical fiber F, the so called *Kato's inequality* compares the trace of the heat operator on (M',g') with the one of the trivial fibration with the same typical fiber F. P. Bérard and S. Gallot [1] and in a different way G. Besson [4] show that, if f is a Riemannian submersion of compact boundaryless manifolds, whose fibers are totally geodesic submanifolds of M', then

$$Z_{M'}(t) \le Z_{M\times F}(t) = Z_M(t)\cdot Z_F(t); \tag{1.3}$$

in particular, if f is a regular ℓ-sheeted Riemannian covering, one obtains:

$$Z_{M'}(t) \le \ell\cdot Z_M(t) \tag{1.4}$$

(see also [22]); they also show that the inequality in (1.3) is an equality if and only if f is the trivial fibration. The inequality (1.4) was extended by J. Tysk [30] to a branched covering whose singularity set is a submanifold of M' of codimension at least 2.

In Section 2, we consider any mapping $f\colon (M',g') \to (M,g)$ which is locally isometric. In this case, Fubini's property is automatically satisfied. Most of the difficulties come from the fact that we don't assume that f is a covering map. A typical example is given by the exponential map, which is a local isometry on an open set in the tangent space (endowed with the pull-back metric), but not a covering. We show (Prop. 2.4) that the series $\sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y')$ converges in the sense of distributions, and that its limit is not greater than $p_M(t,f(x'),y)$.

To obtain the equality case for boundaryless manifolds, we must assume that the manifolds are stochastically complete (Prop. 2.12); in this case, the sum of the series does not depend on $x'\in M'$ but only on $f(x')\in M$. Remember that a Riemannian manifold (X,g) is stochastically complete if and only if $\int_X p_X(t,x,y)\,dv_g(y) = 1$ for any $x\in X$ and for any $t>0$. A geometrical sufficient condition on a complete manifold (X,g) to be stochastically complete concernes the volume of geodesic balls (Grigor'yan theorem 2.9, see [20] and [21] for the proof).

In the case of manifolds with boundary, notice that, when M' has a nonempty boundary, the fact that M' is complete does not imply that M' is geodesically complete. In this case we show (Lemmas 2.2 and 2.3) a weak Hopf-Rinow theorem, and we prove that the restriction of f to the interior of M' is a covering map onto the interior of M. Proposition 2.4 also gives a sharp lower bound of the Dirichlet heat kernel p_M^D in terms of sums of $p_{M'}^D$. To obtain the equality case for manifolds with boundaries (Prop. 2.15), we must assume that f maps the boundary of M' onto the boundary of M, that M' is a complete metric space and that it satisfies the condition of Grigor'yan theorem 2.9.

When f is a ℓ-sheeted Riemannian covering of compact manifolds, we obtain (Cor. 2.18) a first improvement of Kato's inequality (1.4), in which appears explicitely the difference between $\ell\cdot Z_M(t)$ and $Z_{M'}(t)$. We obtain also a comparison between the heat kernels of M' and M in the case where f is not a covering map and, as a typical example, when f is the exponential map (Prop. 2.20, 2.22); this gives an estimate of the heat kernel of a manifold in terms of a computable euclidean one.

It is well known that the heat kernel $p_{M_K}(t, x_0, .)$ of the space-form (M_K, g_K) of constant curvature K only depends on t and on the distance from x_0. There are many works where a pointwise comparison between the heat kernel of a manifold (M, g) and the heat kernel of (M_K, g_K) is established, under suitable assumptions on the curvature of (M, g) (see for instance J. Cheeger and S.T. Yau [10], A. Debiard, B. Gaveau, E. Mazet [15], G. Courtois [11]). We give (Theorem 2.25) a unified proof of these results by clarifying the role played by the different singularities of the Laplacian of the functions which is obtained by transplantation of the heat kernel of M_K, extended by a constant outside a ball. By combining these results with the ones of Section 2, we obtain an effective improvement of Kato's inequality in the Corollaries 2.26, 2.27. The inequalities which appear in the corollaries are sharp (they are for example equalities in the case of the 2-sheeted covering of the real projective space by the standard sphere). These inequalities remain valid when the fibers have infinite cardinality.

In the case that $f:(M', g') \to (M, g)$ is a Riemannian submersion with minimal fibers (the manifolds are now assumed to be compact and boundaryless), we obtain that the resolvent and heat operators on M dominate the resolvent and heat operators on M' respectively (Prop. 3.6). To prove this result, we show that the mapping ϖ from $H_1(M')$ in $H_1(M)$ which sends u on ϖu, where $\varpi u(x)$ is the L^2-norm of u on F_x, is a symmetrization in the sense of G. Besson [5], which obeys a Kato-type inequality with respect to the Laplacians (Def. 3.2): then a generalized Beurling-Deny principle (3.3) gives the result.

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2. Kato's inequality for local (quasi) isometries and applications

a) Some topological remarks.

Let $f:X' \to X$ be any local homeomorphism from a Hausdorff topological space X' to a topological space X. *The unique lift lemma* is then valid, in the sense that *the continuous lift* passing through some point of X' of any continuous mapping $\gamma:Y \to X$, where Y is a connected topological space, *when it exists, is unique.* The proof is the classical one: let $c_1, c_2:Y \to X'$ be two continuous mappings satisfying $c_1(y_0) = c_2(y_0)$ for some $y_0 \in Y$ and $f \circ c_1 = \gamma = f \circ c_2$. The set of $y \in Y$ such that $c_1(y) = c_2(y)$ is closed and open because f is locally injective and X' is Hausdorff.

If f is such that any continuous path $\gamma:[0,1] \to X$ admits a continuous lift $c:[0,1] \to X'$ beginning at any $x' \in f^{-1}(\gamma(0))$, and if X is arcwise connected, then all the fibers $f^{-1}(x)$ have the same cardinality: for $x_1, x_2 \in X$, let us *fix* a path γ_0 from x_1 to x_2. The mapping $f^{-1}(x_1) \to f^{-1}(x_2)$, which sends