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LINEARIZATION OF ANALYTIC AND NON-ANALYTIC GERMS OF DIFFEOMORPHISMS OF $(\mathbb{C}, 0)$

BY TIMOTEO CARLETTI AND STEFANO MARMI (*)

ABSTRACT. — We study Siegel's center problem on the linearization of germs of diffeomorphisms in one variable. In addition of the classical problems of formal and analytic linearization, we give sufficient conditions for the linearization to belong to some algebras of ultradifferentiable germs closed under composition and derivation, including Gevrey classes. In the analytic case we give a positive answer to a question of J.-C. Yoccoz on the optimality of the estimates obtained by the classical majorant series method. In the ultradifferentiable case we prove that the Brjuno condition is sufficient for the linearization to belong to the same class of the germ. If one allows the linearization to be less regular than the germ one finds new arithmetical conditions, weaker than the Brjuno condition. We briefly discuss the optimality of our results.

RÉSUMÉ. — LINÉARISATION DE GERMES DE DIFFÉOMORPHISMES ANALYTIQUES ET NON ANALYTIQUES DE $(\mathbb{C}, 0)$. — Nous étudions le problème du centre de Siegel sur la linéarisabilité des germes de difféomorphismes d'une variable. Aux problèmes classiques de linéarisation formelle et analytique nous ajoutons des conditions suffisantes pour que la linéarisation appartienne à certaines algèbres de germes ultradifférentiables qui sont fermées par composition et dérivation et qui incluent les classes de Gevrey. Dans le cas analytique nous donnons une réponse positive à une question posée par J.-C. Yoccoz sur l'optimalité des estimations obtenues par la méthode classique des séries majorantes. Dans le cas ultradifférentiable nous prouvons que la condition de Brjuno est suffisante pour que la linéarisation appartienne à la même classe que le germe. Si on admet que la linéarisation soit moins regulière que le germe on trouve des nouvelles conditions arithmétiques, plus faibles que la condition de Brjuno. Nous donnons une courte discussion de l'optimalité des résultats obtenus.

1. Introduction

In this paper we study the Siegel center problem [He]. Consider two subalgebras $A_1 \subset A_2$ of $z\mathbb{C}[[z]]$ closed with respect to the composition of formal series. For example $z\mathbb{C}[[z]]$, $z\mathbb{C}\{z\}$ (the usual analytic case) or Gevrey-s classes,

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$s > 0$ (*i.e.* series $F(z) = \sum_{n \geq 0} f_n z^n$ such that there exist $c_1, c_2 > 0$ such that $|f_n| \leq c_1 c_2^n (n!)^s$ for all $n \geq 0$).

Let $F \in A_1$ being such that $F'(0) = \lambda \in \mathbb{C}^*$. We say that F is *linearizable* in A_2 if there exists $H \in A_2$ tangent to the identity and such that

$$F \circ H = H \circ R_\lambda \tag{1.1}$$

where $R_\lambda(z) = \lambda z$.

- When $|\lambda| \neq 1$, the Poincaré-Konigs linearization theorem assures that F is linearizable in A_2.
- When $|\lambda| = 1$, $\lambda = \mathrm{e}^{2\pi i\omega}$, the problem is much more difficult, especially if one looks for *necessary and sufficient* conditions on λ which assure that *all* $F \in A_1$ *with the same* λ *are linearizable in* A_2. The only trivial case is $A_2 = z\mathbb{C}[[z]]$ (formal linearization) for which one only needs to assume that λ is not a root of unity, *i.e.* $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

In the analytic case

$$A_1 = A_2 = z\mathbb{C}\{z\}$$

let S_λ denote the space of analytic germs $F \in z\mathbb{C}\{z\}$ analytic and injective in the unit disk $\mathbb{D}$ and such that $DF(0) = \lambda$ (note that any $F \in z\mathbb{C}\{z\}$ tangent to R_λ may be assumed to belong to S_λ provided that the variable z is suitably rescaled). Let $R(F)$ denote the radius of convergence of the unique tangent to the identity linearization H associated to F. J.-C. Yoccoz [Yo] proved that the *Brjuno condition* (see Appendix A) is necessary and sufficient for having $R(F) > 0$ for all $F \in S_\lambda$. More precisely Yoccoz proved the following estimate: assume that $\lambda = \mathrm{e}^{2\pi i\omega}$ is a Brjuno number. There exists a universal constant $C > 0$ (independent of λ) such that

$$\left|\log R(\omega) + B(\omega)\right| \leq C$$

where $R(\omega) = \inf_{F \in S_\lambda} R(F)$ and B is the Brjuno function (A.3). Thus

$$\log R(\omega) \geq -B(\omega) - C.$$

Brjuno's proof [Br] gives an estimate of the form

$$\log r(\omega) \geq -C'B(\omega) - C''$$

where one can choose $C' = 2$ (see [He]). Yoccoz's proof is based on a geometric renormalization argument and Yoccoz himself asked whether or not was possible to obtain $C' = 1$ by direct manipulation of the power series expansion of the linearization H as in Brjuno's proof (see [Yo, rem. 2.7.1, p. 21]). Using an arithmetical lemma due to Davie [Da] (Appendix B) we give a positive answer (Theorem 2.1) to Yoccoz's question.

We then consider the more general ultradifferentiable case

$$A_1 \subset A_2 \neq z\mathbb{C}\{z\}.$$

If one requires $A_2 = A_1$, *i.e.* the linearization H to be as regular as the given germ F, once again the Brjuno condition is sufficient. Our methods do not allow us to conclude that the Brjuno condition is also necessary, a statement which is in general false as we show in section 2.3 where we exhibit a Gevrey-like class for which the sufficient condition coincides with the optimal arithmetical condition for the associated linear problem. Nevertheless it is quite interesting to notice that given any algebra of formal power series which is closed under composition (as it should if one whishes to study conjugacy problems) and derivation a germ in the algebra is linearizable *in the same algebra* if the Brjuno condition is satisfied.

If the linearization is allowed to be less regular than the given germ (*i.e.* A_1 is a proper subset of A_2) one finds a new arithmetical condition, weaker than the Brjuno condition. This condition is also optimal if the small divisors are replaced with their absolute values as we show in Section 2.4. We discuss two examples, including Gevrey-s classes.[1]

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2. The Siegel center problem

Our first step will be the formal solution of equation (1.1) assuming only that $F \in z\mathbb{C}[[z]]$. Since $F \in z\mathbb{C}\,[[z]]$ is assumed to be tangent to R_λ then

$$F(z) = \sum_{n\geq 1} f_n z^n$$

with $f_1 = \lambda$. Analogously since $H \in z\mathbb{C}\,[[z]]$ is tangent to the identity

$$H(z) = \sum_{n=1}^{\infty} h_n z^n$$

with $h_1 = 1$. If λ is not a root of unity equation (1.1) has a unique solution $H \in z\mathbb{C}\,[[z]]$ tangent to the identity: the power series coefficients satisfy the recurrence relation

$$h_1 = 1, \quad h_n = \frac{1}{\lambda^n - \lambda} \sum_{m=2}^{n} f_m \sum_{\substack{n_1+\cdots+n_m=n \\ n_i\geq 1}} h_{n_1}\cdots h_{n_m}. \tag{2.1}$$

[1] We refer the reader interested in small divisors and Gevrey-s classes to [Lo], [GY1], [GY2].

In [Ca] it is shown how to generalize the classical Lagrange inversion formula to non-analytic inversion problems on the field of formal power series so as to obtain an explicit non-recursive formula for the power series coefficients of H.

2.1. The analytic case: a direct proof of Yoccoz's lower bound.

Let S_λ denote the space of germs $F \in z\mathbb{C}\{z\}$ analytic and injective in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\,,\, |z| < 1\}$ such that $DF(0) = \lambda$ and assume that $\lambda = e^{2\pi i\omega}$ with $\omega \in \mathbb{R} \setminus \mathbb{Q}$. With the topology of uniform convergence on compact subsets of $\mathbb{D}$, S_λ is a compact space. Let $H_F \in z\mathbb{C}[[z]]$ denote the unique tangent to the identity formal linearization associated to F, *i.e.* the unique formal solution of (1.1). Its power series coefficients are given by (2.1). Let $R(F)$ denote the radius of convergence of H_F. Following Yoccoz [Yo, p. 20] we define

$$R(\omega) = \inf_{F \in S_\lambda} R(F).$$

We will prove the following

THEOREM 2.1 (Yoccoz's lower bound). — *One has*

$$\log R(\omega) \geq -B(\omega) - C \tag{2.2}$$

where C is a universal constant (independent of ω) and B is the Brjuno function (A.3).

Our method of proof of Theorem 2.1 will be to apply an arithmetical lemma due to Davie (see Appendix B) to estimate the sm ıll divisors contribution to (2.1). This is actually a variation of the classical majorant series method as used in [Si] and [Br].

Proof. — Let $s(z) = \sum_{n\geq 1} s_n z^n$ be the unique solution analytic at $z = 0$ of the equation

$$s(z) = z + \sigma\big(s(z)\big),$$

where

$$\sigma(z) = \frac{z^2(2-z)}{(1-z)^2} = \sum_{n\geq 2} n z^n.$$

The coefficients satisfy

$$s_1 = 1, \quad s_n = \sum_{m=2}^{n} m \sum_{\substack{n_1+\cdots+n_m=n \\ n_i\geq 1}} s_{n_1}\cdots s_{n_m}. \tag{2.3}$$

Clearly there exist two positive constants γ_1, γ_2 such that

$$|s_n| \leq \gamma_1 \gamma_2^n. \tag{2.4}$$