Bull. Soc. math. France
129 (2), 2001, p. 285–316

PROFILE DECOMPOSITION FOR SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

by Isabelle Gallagher

Abstract. — We consider sequences of solutions of the Navier-Stokes equations in $\mathbb{R}^3$, associated with sequences of initial data bounded in $\dot{H}^{1/2}$. We prove, in the spirit of the work of H. Bahouri and P. Gérard (in the case of the wave equation), that they can be decomposed into a sum of orthogonal profiles, bounded in $\dot{H}^{1/2}$, up to a remainder term small in L^3; the method is based on the proof of a similar result for the heat equation, followed by a perturbation–type argument. If $\mathcal{A}$ is an "admissible" space (in particular L^3, $\dot{B}^{-1+3/p}_{p,\infty}$ for $p < +\infty$ or ∇BMO), and if $\mathcal{B}^{\mathcal{A}}_{NS}$ is the largest ball in $\mathcal{A}$ centered at zero such that the elements of $\dot{H}^{1/2} \cap \mathcal{B}^{\mathcal{A}}_{NS}$ generate global solutions, then we obtain as a corollary an *a priori* estimate for those solutions. We also prove that the mapping from data in $\dot{H}^{1/2} \cap \mathcal{B}^{\mathcal{A}}_{NS}$ to the associate solution is Lipschitz.

Texte reçu le 6 juillet 2000, accepté le 13 octobre 2000

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2000 Mathematics Subject Classification. — 35B45, 35Q30, 76D05.

Key words and phrases. — Navier–Stokes, explosion, profiles, a priori estimate, admissible space.

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RÉSUMÉ (*Décomposition en profils pour les solutions des équations de Navier-Stokes*)
On considère des suites de solutions des équations de Navier–Stokes dans $\mathbb{R}^3$, associées à des suites de données initiales bornées dans $\dot H^{1/2}$. On montre, dans l'esprit du travail de H. Bahouri et P. Gérard (dans le cas de l'équation des ondes), qu'elles peuvent être décomposées en une somme de profils orthogonaux, bornés dans $\dot H^{1/2}$, à un terme de reste près, petit dans L^3 ; la méthode s'appuie sur la démonstration d'un résultat analogue pour l'équation de la chaleur, suivi d'un argument de perturbation. Si $\mathcal{A}$ est un espace « admissible » (en particulier L^3, $\dot B_{p,\infty}^{-1+3/p}$ pour $p < +\infty$ ou ∇BMO), et si $\mathcal{B}_{NS}^{\mathcal{A}}$ est la plus grande boule de de $\mathcal{A}$ centrée en zéro, telle que les éléments de $\dot H^{1/2} \cap \mathcal{B}_{NS}^{\mathcal{A}}$ génèrent des solutions globales, alors on obtient en corollaire une estimation *a priori* pour ces solutions. On montre aussi que l'application associant une donnée dans $\dot H^{1/2} \cap \mathcal{B}_{NS}^{\mathcal{A}}$ à sa solution est lipschitzienne.

1. Introduction

We are interested in the incompressible Navier-Stokes equations in three space dimensions

$$
\text{(NS)} \qquad \begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v = -\nabla p \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^3,\\ \operatorname{div} v = 0 \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^3,\\ v_{|t=0} = v_0, \end{cases}
$$

where v_0 is a divergence free vector field, $v(t,x)$ and $p(t,x)$ are respectively the velocity and the pressure fields of the fluid, and $\nu > 0$ is the viscosity. The velocity is a three-component vector field, and the pressure is a scalar field. The divergence free condition on v represents the incompressibility of the fluid. Here t and x are respectively the time and the space variables, with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^3$. All the results stated here hold in the more general case of $x \in \mathbb{R}^d, d \geq 3$, with obvious adaptations, namely in the orders of the functional spaces considered.

In order to motivate our study, let us recall a few well–known facts concerning the system (NS). The most important results about the Cauchy problem were obtained by J. Leray in [21], who proved that for divergence free data $v_0 \in L^2(\mathbb{R}^3)$, there is a global, weak solution v of (NS) with

$$
v \in L^\infty\big(\mathbb{R}^+, L^2(\mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}^+, \dot H^1(\mathbb{R}^3)\big),
$$

where $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space of order p, and where we have noted $\dot H^s(\mathbb{R}^3)$ the homogeneous Sobolev space of order s, defined by

$$
\forall s < \frac{3}{2}, \quad \dot H^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \Big\{ u \in \mathcal{S}'(\mathbb{R}^3);\ \|u\|_{\dot H^s(\mathbb{R}^3)} < +\infty \Big\},
$$

where

$$
\|u\|_{\dot H^s(\mathbb{R}^3)} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^3} |\xi|^{2s} \left|\widehat u(\xi)\right|^2 \mathrm{d}\xi \Big)^{1/2},
$$

and $\widehat{u}$ is the Fourier transform of u. We will note $(\cdot\,|\,\cdot)_{\dot H^s(\mathbb{R}^3)}$ the scalar product in $\dot H^s(\mathbb{R}^3)$. The restriction $s < \frac{3}{2}$ is for $\|u\|_{\dot H^s(\mathbb{R}^3)}$ to be a norm and not a semi-norm. Note that the inhomogeneous Sobolev space $H^s(\mathbb{R}^3)$ is of course defined in the same way, where $|\xi|^{2s}$ is replaced by $(1+|\xi|^2)^s$. In the following, we will call $\dot H^{3/2}(\mathbb{R}^3)$ the space of vector fields whose components have first derivatives in $\dot H^{1/2}(\mathbb{R}^3)$.

The solutions constructed by J. Leray satisfy moreover the energy inequality

$$\forall t \geq 0, \quad \big\|v(t)\big\|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_0^t \big\|\nabla v(s)\big\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}s \leq \|v_0\|^2_{L^2(\mathbb{R}^3)}. \tag{1.1}$$

Those solutions are not known to be unique (except in two space dimensions); many studies exist on that problem of uniqueness, and the starting point of our study will be the result of H. Fujita and T. Kato [8]. It can be stated in the following way (see [4] for instance): if v_0 is in $\dot H^{1/2}(\mathbb{R}^3)$, then there exists a unique maximal time $T_* > 0$ and a unique solution v associated with v_0 such that

$$v \in C^0\big([0,T], \dot H^{1/2}(\mathbb{R}^3)\big) \cap L^2\big([0,T], \dot H^{3/2}(\mathbb{R}^3)\big) \quad \text{for all } T < T_*.$$

Moreover, if $T_* < +\infty$, then we have

$$\lim_{T\to T_*} \|v\|_{L^2([0,T],\dot H^{3/2}(\mathbb{R}^3))} = +\infty. \tag{1.2}$$

Furthermore, there exists a universal constant c such that

$$\|v_0\|_{\dot H^{1/2}(\mathbb{R}^3)} \leq c\nu \quad \Longrightarrow \quad T_* = +\infty, \tag{1.3}$$

and we have in that case, for any $t \geq 0$,

$$\big\|v(t)\big\|^2_{\dot H^{1/2}(\mathbb{R}^3)} + \nu \int_0^t \big\|v(s)\big\|^2_{\dot H^{3/2}(\mathbb{R}^3)} \mathrm{d}s \leq \|v_0\|^2_{\dot H^{1/2}(\mathbb{R}^3)}. \tag{1.4}$$

Finally it is well known (see for instance [21] or [7], Remark 10.3 (a)) that we have the following weak-strong uniqueness result:

$$\forall\, v_0 \in L^2 \cap \dot H^{1/2}(\mathbb{R}^3), \quad NS(v_0) \quad \text{satisfies} \quad (1.1), \tag{1.5}$$

where, as in the whole of this text, we have noted $NS(v_0)$ the unique solution of (NS) associated with the initial data $v_0 \in \dot H^{1/2}(\mathbb{R}^3)$.

One important aspect to keep in mind in the study of (NS) is the scaling of the equation. It is easy to check the following property: for any real number λ,

$$v = NS(v_0) \quad \Longleftrightarrow \quad v_\lambda = NS(v_{0,\lambda}), \tag{1.6}$$

with

$$v_\lambda(t,x) \stackrel{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x) \quad \text{and} \quad v_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda v_0(\lambda x).$$

Note that the $\dot H^{1/2}(\mathbb{R}^3)$ norm is clearly conserved under the transformation $v_0 \mapsto v_{0,\lambda}$. Many existence and uniqueness results have been obtained for data in such function spaces, invariant under that transformation; it is impossible to present here all the function spaces in which such results have been obtained, so let us simply recall the chain of spaces

$$\dot{H}^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \subset \dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)_{|p<+\infty} \subset \nabla BMO(\mathbb{R}^3) \subset \dot{C}^{-1}(\mathbb{R}^3).$$

In that chain of spaces, $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ stands for a homogeneous Besov space. We shall not be using those spaces explicitly in this paper, so we will merely recall the following definition, using Littlewood Paley theory, and we refer for instance to [5] for a detailed presentation of the theory, and to [22] or [25] for the analysis of Besov spaces: elements of $\dot{B}^{s}_{p,\infty}(\mathbb{R}^3)$ satisfy

$$\|u\|_{\dot{B}^{s}_{p,\infty}(\mathbb{R}^3)} \stackrel{\text{def}}{=} \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^3)} < +\infty,$$

where Δ_j is a Littlewood-Paley operator, defined by

$$\widehat{\Delta_j u}(\xi) \stackrel{\text{def}}{=} \varphi\big(2^{-j}|\xi|\big)\,\widehat{u}(\xi)$$

and $\varphi \in \mathcal{C}^\infty_c([\frac{1}{2},2])$ satisfies $\sum_{j\in\mathbb{Z}}\varphi(2^{-j}t) = 1$, for all $t>0$.

Furthermore, $\nabla BMO(\mathbb{R}^3)$ stands for the space of functions which are first derivatives of functions in $BMO(\mathbb{R}^3)$. We recall below the definition of the norm $\|u\|_{BMO(\mathbb{R}^3)}$, and refer to [24] for a detailed presentation of that space:

$$\|u\|_{BMO(\mathbb{R}^3)} \stackrel{\text{def}}{=} \sup_{x_0,R}\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} |u-u_{B(x_0,R)}|\,\mathrm{d}x,$$

where

$$u_{B(x_0,R)} \stackrel{\text{def}}{=} \frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} u(x)\,\mathrm{d}x.$$

In all those spaces except for the last, analogous existence and uniqueness theorems to the case $\dot{H}^{1/2}(\mathbb{R}^3)$ have been proved. We refer respectively to T. Kato [18] and G. Furioli, P.-G. Lemarié and E. Terraneo [10] for the proof of the $L^3(\mathbb{R}^3)$ case, to the book of M. Cannone [3] and the work of F. Planchon [23] for $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3), p<+\infty$, and finally to H. Koch and D. Tataru [20] for the space ∇BMO. In the space $\dot{C}^{-1}(\mathbb{R}^3) \stackrel{\text{def}}{=} \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, uniqueness was proved by J.-Y. Chemin in [6], supposing the data is also in the energy space $L^2(\mathbb{R}^3)$.

In relation with the result of H. Fujita and T. Kato mentionned above, let us give the following definitions: we define the function spaces

$$\begin{cases} E_T \stackrel{\text{def}}{=} C^0\big([0,T],\dot{H}^{1/2}(\mathbb{R}^3)\big)\cap L^2\big([0,T],\dot{H}^{3/2}(\mathbb{R}^3)\big),\\ E_\infty \stackrel{\text{def}}{=} C^0_b\big(\mathbb{R}^+,\dot{H}^{1/2}(\mathbb{R}^3)\big)\cap L^2\big(\mathbb{R}^+,\dot{H}^{3/2}(\mathbb{R}^3)\big),\end{cases} \tag{1.7}$$

where C^0_b denotes the set of bounded, continuous functions; we also define the sets of initial data yielding solutions of (NS) in E_T and E_∞ respectively,

$$\mathcal{D}_T \stackrel{\text{def}}{=} \big\{v_0\in\dot{H}^{1/2}(\mathbb{R}^3)\mid NS(v_0)\in E_T\big\},$$
$$\mathcal{D}_\infty \stackrel{\text{def}}{=} \big\{v_0\in\dot{H}^{1/2}(\mathbb{R}^3)\mid NS(v_0)\in E_\infty\big\}.$$

Finally we define, for any vector field v,

$$
(1.8)\qquad \begin{cases} \|v\|_{E^\nu_T} \stackrel{\text{def}}{=} \sup_{t\leq T}\big(\|v(t)\|^2_{\dot H^{1/2}(\mathbb{R}^3)} + 2\nu\,\|v\|^2_{L^2([0,t],\dot H^{3/2}(\mathbb{R}^3))}\big)^{1/2}, \\ \|v\|_{E^\nu_\infty} \stackrel{\text{def}}{=} \big(\|v\|^2_{L^\infty(\mathbb{R}^+,\dot H^{1/2}(\mathbb{R}^3))} + 2\nu\,\|v\|^2_{L^2(\mathbb{R}^+,\dot H^{3/2}(\mathbb{R}^3))}\big)^{1/2}. \end{cases}
$$

Remark. — Note that nothing prevents *a priori* the life span T_* associated with some data v_0 to satisfy $T_* = +\infty$ with $v_0 \notin \mathcal{D}_\infty$: in that case,

$$
\lim_{T\to+\infty} \big\|NS(v_0)\big\|_{L^2([0,T],\dot H^{3/2}(\mathbb{R}^3))} = +\infty.
$$

Definition 1. — *Let $\mathcal{A} \subset \mathcal{S}'(\mathbb{R}^3)$ be a Banach space such that the embedding $\dot H^{1/2}(\mathbb{R}^3) \subset \mathcal{A}$ is continuous. Then $\mathcal{A}$ is admissible if and only if the following properties hold:*

(i) *The norm $\|\ \|_{\mathcal{A}}$ is invariant under the transformations*

$$
\varphi \longmapsto \lambda\varphi(\lambda\cdot) \quad \forall\lambda\in\mathbb{R} \quad \textit{and} \quad \varphi\mapsto\varphi(\cdot - x_0) \quad \forall x_0\in\mathbb{R}^3.
$$

(ii) *There exists a constant $c^{\mathcal{A}}_\nu$ depending only on ν and $\mathcal{A}$ such that if φ is an element of $\dot H^{1/2}(\mathbb{R}^3)$ and $\|\varphi\|_{\mathcal{A}}$ is smaller than $c^{\mathcal{A}}_\nu$, then φ is in $\mathcal{D}_\infty$.*

Example 1. — An obvious example is of course $\dot H^{1/2}(\mathbb{R}^3)$; point (i) is clear, and point (ii) is due to H. Fujita and T. Kato's theorem recalled above.

Example 2. — Similarly $L^3(\mathbb{R}^3)$ satisfies point (i), and a proof of (ii) can be found in Proposition A.1 in the Appendix.

Example 3. — The Besov space $\dot B^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ satisfies point (i), and point (ii) is proved for instance in Theorem 3.4.2 of [3] for $p < +\infty$.

Example 4. — The space $\nabla BMO(\mathbb{R}^3)$ is a Banach space satisfying points (i) and (ii), as proved in [9].

In the following, for any admissible space $\mathcal{A}$ in the sense of Definition 1, we shall define the constant $C^{\mathcal{A}}_{NS} \in \mathbb{R}^+\cup\{+\infty\}$ by

$$
(1.9)\qquad C^{\mathcal{A}}_{NS} \stackrel{\text{def}}{=} \sup\big\{\rho>0\,;\ \overline{\mathcal{B}}^{\mathcal{A}}_\rho \cap \dot H^{1/2}(\mathbb{R}^3) \subset \mathcal{D}_\infty\big\},
$$

where

$$
\mathcal{B}^{\mathcal{A}}_\rho \stackrel{\text{def}}{=} \big\{\varphi\in\mathcal{A}\,;\ \|\varphi\|_{\mathcal{A}} < \rho\big\}
$$

and we will note

$$
(1.10)\qquad \mathcal{B}^{\mathcal{A}}_{NS} \stackrel{\text{def}}{=} \mathcal{B}^{\mathcal{A}}_{C_{NS}}.
$$

In other words, the set $\mathcal{B}^{\mathcal{A}}_{NS}$ is the largest ball in $\mathcal{A}$ whose intersection with $\dot H^{1/2}(\mathbb{R}^3)$ is a subset of $\mathcal{D}_\infty$. Note that we obviously have $C^{\mathcal{A}}_{NS} \geq c^{\mathcal{A}}_\nu$, where $c^{\mathcal{A}}_\nu$ was defined in Property (ii) of Definition 1. The following result will be proved in the Appendix.