Bull. Soc. math. France
129 (4), 2001, p. 547–591

ON THE PARTIAL ALGEBRAICITY OF HOLOMORPHIC MAPPINGS BETWEEN TWO REAL ALGEBRAIC SETS

BY JOËL MERKER

ABSTRACT. — The rigidity properties of the local invariants of real algebraic Cauchy-Riemann structures imposes upon holomorphic mappings some global rational properties (Poincaré 1907) or more generally algebraic ones (Webster 1977). Our principal goal will be to unify the classical or recent results in the subject, building on a study of the transcendence degree, to discuss also the usual assumption of minimality in the sense of Tumanov, in arbitrary dimension, without rank assumption and for holomorphic mappings between two arbitrary real algebraic sets.

RÉSUMÉ (*Algébricité partielle des applications holomorphes entre deux ensembles algébriques réels*)

La rigidité des invariants locaux des structures de Cauchy-Riemann réelles algébriques impose aux applications holomorphes des propriétés globales de rationalité (Poincaré 1907), ou plus généralement d'algébricité (Webster 1977). Notre objectif principal sera d'unifier les résultats classiques ou récents, grâce à une étude du degré de transcendance, de discuter aussi l'hypothèse habituelle de minimalité au sens de Tumanov, et ce en dimension quelconque, sans hypothèse de rang et pour des applications holomorphes quelconques entre deux ensembles algébriques réels arbitraires.

Texte reçu le 20 décembre 1999, révisé le 4 septembre 2000 et le 30 janvier 2001, accepté le 7 mars 2001

JOËL MERKER, Laboratoire d'Analyse, Topologie et Probabilités, Centre de Mathématiques et d'Informatique, UMR 6632, 39 rue Joliot Curie, 13453 Marseille Cedex 13 (France)
E-mail : `merkercmi.univ-mrs.fr`

2000 Mathematics Subject Classification. — 32V25, 32V40, 32V15, 32V10.

Key words and phrases. — Local holomorphic mappings, Real algebraic sets, Transcendence degree, Local algebraic foliations, Minimality in the sense of Tumanov, Segre chains.

1. Introduction

The algebraicity or the rationality of local holomorphic mappings between real algebraic CR manifolds can be considered to be one of the most remarkable phenomena in CR geometry. Introducing the consideration of Segre varieties in the historical article [17], Webster generalized the classical rationality properties of self-mappings between three-dimensional spheres discovered by Poincaré and later extended by Tanaka to arbitrary dimension. Webster's theorem states that biholomorphisms between Levi non-degenerate real algebraic hypersurfaces in $\mathbb{C}^n$ are algebraic. Around the eighties, some authors studied proper holomorphic mappings between spheres of different dimensions or between pieces of strongly pseudoconvex real algebraic hypersurfaces, notably Pelles, Alexander, Fefferman, Pinchuk, Chern-Moser, Diederich-Fornaess, Faran, Cima-Suffridge, Forstnerič, Sukhov, and others (complete references are provided in [2], [5], [8], [13], [15], [17], [18]). In the past decade, removing the equidimensionality condition in the classical theorem of Webster, Sukhov for mappings between Levi non-degenerate quadrics [15], Huang [8] for mappings between strongly pseudoconvex hypersurfaces, and Sharipov-Sukhov [13] for mappings between general Levi non-degenerate real algebraic CR manifolds have exhibited various sufficient conditions for the algebraicity of a general local holomorphic map $f : M \to M'$ between two real algebraic CR manifolds $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n'}$. A necessary and sufficient condition, but with a rank condition on f is provided in [2]. Recently, using purely algebraic methods, Coupet-Meylan-Sukhov (*cf.* [5]; see also [6]) have estimated the transcendence degree of f directly. Building on their work, we aim essentially to study the algebraicity question in full generality (*cf.* Problem 2.4) and to unify the various approaches of [2], [5], [8], [12], [13], [17], [18]. Notably, we shall state necessary and sufficient conditions for the algebraicity of f without rank condition and we shall study the geometry of the minimality assumption thoroughly.

2. Presentation of the main result

2.1. Algebraicity of holomorphic mappings and their transcendence degree. — Let $U \subset \mathbb{C}^n$ be a small nonempty open polydisc. A holomorphic mapping $f : U \to \mathbb{C}^{n'}$, $f \in \mathcal{H}(U, \mathbb{C}^{n'})$, is called *algebraic* if its graph is contained in an irreducible n-dimensional complex algebraic subset of $\mathbb{C}^n \times \mathbb{C}^{n'}$. Using classical elimination theory, one can show that, equivalently, each of its components $g := f_1, \dots, f_{n'}$ satisfies a nontrivial polynomial equation $g^r a_r + \cdots + a_0 = 0$, the $a_j \in \mathbb{C}[z]$ being polynomials. We recall that a set $\Sigma \subset U$ is called real algebraic if it is given as the zero set in U of a finite number of *real algebraic polynomials* in $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$.

Let us denote by $\mathcal{A} = \mathbb{C}[z]$ the ring of complex polynomial functions over $\mathbb{C}^n$ and by $\mathcal{R} = \mathbb{C}(z)$ its quotient field $\mathrm{Fr}\,(\mathcal{A})$. By $\mathcal{R}(f_1, \dots, f_{n'})$ we understand the field generated by $f_1, \ldots, f_{n'}$ over $\mathcal{R}$, which is a subfield of the field of meromorphic functions over U and which identifies with the collection of rational functions

$$R(f_1, \ldots, f_{n'}) = P(f_1, \ldots, f_{n'})/Q(f_1, \ldots, f_{n'}), \tag{2.2}$$

$P, Q \in \mathcal{R}[x_1, \ldots, x_{n'}]$ and $Q \neq 0$.

Following [5], the *transcendence degree* $\nabla^{\mathrm{tr}}(f)$ of the field $\mathcal{R}(f_1, \ldots, f_{n'})$ with respect to the field $\mathcal{R}$ provides an integer-valued invariant measuring the lack of algebraicity of f. In particular, $\nabla^{\mathrm{tr}}(f)$ *is zero if and only if f is algebraic.*

Indeed, by definition $\nabla^{\mathrm{tr}}(f)$ coincides with the maximal cardinal number κ' of a subset

$$\{f_{j_1}, \ldots, f_{j_{\kappa'}}\} \subset \{f_1, \ldots, f_{n'}\}, \quad 1 \leq j_1 < \cdots < j_{\kappa'} \leq n'$$

which is algebraically independent over $\mathcal{R}$. In other words, $\nabla^{\mathrm{tr}}(f) = \kappa'$ means that there exists a subset $\{f_{j_1}, \ldots, f_{j_{\kappa'}}\} \subset \{f_1, \ldots, f_{n'}\}$ such that there does not exist a nontrivial relation

$$P(f_{j_1}, \ldots, f_{j_{\kappa'}}) \equiv 0 \text{ in } \mathcal{H}(U), \quad P \in \mathcal{R}[x_1, \ldots, x_{\kappa'}] \setminus \{0\},$$

but that for every λ, $\kappa' + 1 \leq \lambda \leq n'$, every $1 \leq j_1 < \cdots < j_\lambda \leq n'$, there exists an algebraic relation $Q(f_{j_1}, \ldots, f_{j_\lambda}) \equiv 0$, $Q \in \mathcal{R}[x_1, \ldots, x_\lambda] \setminus \{0\}$. Of course,

$$\nabla^{\mathrm{tr}}(f_1, \ldots, f_{n'}) \leq n'.$$

An equivalent geometric characterization of $\nabla^{\mathrm{tr}}(f)$ states that $\nabla^{\mathrm{tr}}(f) = \kappa'$ if and only if the *dimension* of the minimal for inclusion *complex algebraic* set $\Lambda_f \subset U \times \mathbb{C}^{n'}$ containing the *graph*

$$\Gamma_f = \big\{(z, f(z)) \in U \times \mathbb{C}^{n'} : z \in U\big\}$$

of f is equal to $n + \kappa'$ (this complex algebraic set Λ_f is of course necessarily irreducible). In other words, $\nabla^{\mathrm{tr}}(f)$ is an invariant intrinsically attached to f which is given with f and which possesses an algebraic *and* a geometric signification. In a metaphoric sense, we can think that $\nabla^{\mathrm{tr}}(f)$ measures the *lack of algebraicity* of f, or conversely, that it provides an estimation of the *maximal partial algebraicity properties* of f.

2.3. Presentation of the main result. — Then, because the transcendence degree is an appropriate invariant, more general than the dichotomy between algebraic and non-algebraic objects, we shall as in [5] study directly the transcendence degree of holomorphic mappings between two real algebraic sets. Our main goal is to provide a synthesis of the results in [2], [5], [8], [13], [17], [18]. Thus, quite generally, let $f : U \to \mathbb{C}^{n'}$ be a local holomorphic mapping sending an arbitrary irreducible real algebraic set $\Sigma \cap U$ into another real algebraic set $\Sigma' \subset \mathbb{C}^{n'}$. As the algebraicity of f is a non-local property, we shall assume in the sequel that $\Sigma \cap U$ is a smooth closed CR-submanifold of U and that there exists a second polydisc $U' \subset \mathbb{C}^{n'}$ with $f(U) \subset U'$ such that $\Sigma' \cap U'$ is also a smoooth closed CR-submanifold of U'. We shall denote by M and M' these connected local CR pieces of Σ in U and of Σ' in U'. Of course, after shrinking again U and U', we can suppose that f is of constant rank over U. The topic of this article is to study in full generality the following problem by seeking an optimal bound:

PROBLEM 2.4. — *Estimate $\nabla^{\rm tr}(f)$ in terms of geometric invariants of f, Σ, Σ'.*

To begin with, we shall first assume that M is somewhere minimal in the sense of Tumanov, as in [2], [5], [8], [13], [17], [18]. In the sequel, we shall say that a property $\mathcal{P}$ *holds at a Zariski-generic point* $p \in \Sigma$ if there exist a proper real algebraic subset E of Σ, depending on $\mathcal{P}$, such that the property $\mathcal{P}$ holds at each point of $\Sigma \setminus E$. Let Δ be the unit disc in $\mathbb{C}$. Our main result lies in the following statement from which we shall recover every algebraicity result of the cited literature.

THEOREM 2.5. — *Assume that M is CR-generic, connected and minimal in the sense of Tumanov at a Zariski-generic point, and let Σ'' be the minimal (for inclusion, hence irreducible) real algebraic set satisfying $f(M) \subset \Sigma'' \subset \Sigma'$. Let κ' denote the transcendence degree $\nabla^{\rm tr}(f)$ of f. Then near a Zariski-generic point $p'' \in \Sigma''$, there exists a local algebraic coordinate system in which Σ'' is of the form $\Delta^{\kappa'} \times \underline{\Sigma}''$ for some real algebraic variety $\underline{\Sigma}'' \subset \mathbb{C}^{n'-\kappa'}$.*

This theorem says that the degree of nonalgebraicity of f imposes some *degeneracy condition* on Σ', namely to contain a smaller real algebraic set Σ'' which is 'degenerate' in the sense that it can be locally straightened to be a product by a complex $\nabla^{\rm tr}(f)$-dimensional polydisc at almost every point. The main interest of Theorem 2.5 lies in fact in its various reciprocal forms which are listed in §3 below. Of course, the assumption that $\nabla^{\rm tr}(f)$ equals an integer κ' is no assumption at all, since $\nabla^{\rm tr}(f)$ is automatically given with f, but in truth abstractly, namely in a non-constructive way, as is Σ''. The only unjustified assumption with respect to Problem 2.4 is the minimality of M in the sense of Tumanov and it remains also to explain why the estimate given by Theorem 2.5 is sharp and satisfactory.

Thus, let us firstly consider the sharpness. If Σ'' is an irreducible real algebraic set as above, it can be shown that there exists the largest integer $\kappa_{\Sigma''}$ such that Σ'' is a product of the form $\Delta^{\kappa_{\Sigma''}} \times \underline{\Sigma}''$ near a Zariski-generic point in suitable local algebraic coordinates (see Theorem 9.10). This integer will also be abbreviated by $\kappa_{\Sigma''}$ and we shall say that Σ'' is $\kappa_{\Sigma''}$*-algebraically degenerate.* We shall also write

$$\Sigma'' \cap V'' \cong_{\mathcal{A}} \Delta^{\kappa_{\Sigma''}} \times \underline{\Sigma}''$$

to mean that Σ'' intersected with the small open set V'' is equivalent to a product in *complex algebraic* (abbreviation: $\mathcal{A}$) coordinates. Then Theorem 2.5

states in summary that $\kappa_{\Sigma''} \geq \nabla^{\rm tr}(f)$. With this notion defined and this rephrasing of Theorem 2.5 at hand, we now notice that it can of course well happen that $\kappa_{\Sigma''} > \nabla^{\rm tr}(f)$. For instance, this happens when $n = n'$, when f is an algebraic biholomorphic map, so $\nabla^{\rm tr}(f) = 0$, and when $\Sigma = \Sigma' = \Sigma'' = \mathbb{C}^n$ simply. So what? In case where $\kappa_{\Sigma''} > \nabla^{\rm tr}(f)$, by an elementary observation we shall show that *a suitable perturbation of f can raise and maximize its possible transcendence degree*. The precise statement, which establishes the desired sharpness, is as follows.

THEOREM 2.6. — *Assume that f is nonconstant and that Σ'' is the* minimal for inclusion *real algebraic set satisfying $f(M) \subset \Sigma'' \subset \Sigma'$. Remember that Σ'' is locally equivalent to $\Delta^{\kappa_{\Sigma''}} \times \underline{\Sigma}''$ at a Zariski-generic point. Then there exist a point $p \in M$ such that $f(p) =: p'' \in \Sigma''$ is a Zariski-generic point of Σ'' and a local holomorphic self-map ϕ of Σ'' fixing p'' which is arbitrarily close to the identity map, such that $\nabla^{\rm tr}(\phi \circ f) = \kappa_{\Sigma''}$ (of course, because of Theorem* 2.5*, it is impossible to produce $\nabla^{\rm tr}(\phi \circ f) > \kappa_{\Sigma''}$).*

Secondly, let us discuss the (until now still unjustified) minimality in the sense of Tumanov assumption. Remember that CR manifold without infinitesimal CR automorphisms are quite poor, since they carry few self-maps. In §13.2 below, we shall observe the following.

THEOREM 2.7. — *Let M be a nowhere minimal real algebraic CR-generic manifold and assume that the space of infinitesimal CR-automorphisms of M is nonzero. Then M admits a local one-parameter family of nonalgebraic biholomorphic self-maps.*

3. Five corollaries

The direct converse of the main Theorem 2.5 bounds $\nabla^{\rm tr}(f)$ as follows and gives an optimal sufficient condition for f to be algebraic.

THEOREM 3.1. — *Let $f \in \mathcal{H}(U, \mathbb{C}^{n'})$ with $f(M) \subset \Sigma'$ and assume M is CR-generic and minimal at a Zariski-generic point. Then $\nabla^{\rm tr}(f) \leq \kappa_{\Sigma''} =$ algebraic degeneracy degree of the minimal for inclusion real algebraic set Σ'' such that $f(M) \subset \Sigma'' \subset \Sigma'$. In particular, f is necessarily algebraic if there does not exist an* 1*-algebraically degenerate real algebraic set Σ''' with $f(M) \subset \Sigma''' \subset \Sigma'$.*

As Theorem 2.6 shows that, otherwise, f can be slightly perturbed to be nonalgebraic, this theorem provides a *necessary and sufficient* condition for f to be algebraic. Thanks to this synthetic general converse, we will recover results of the cited litterature as corollaries. We also obtain as a corollary the celebrated algebraicity result in [17] from the following statement.