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ON THE SIZE OF THE SETS OF GRADIENTS OF BUMP FUNCTIONS AND STARLIKE BODIES ON THE HILBERT SPACE

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Abstract. — We study the size of the sets of gradients of bump functions on the Hilbert space ℓ_2, and the related question as to how small the set of tangent hyperplanes to a smooth bounded starlike body in ℓ_2 can be. We find that those sets can be quite small. On the one hand, the usual norm of the Hilbert space ℓ_2 can be uniformly approximated by C^1 smooth Lipschitz functions ψ so that the cones generated by the ranges of its derivatives $\psi'(\ell_2)$ have empty interior. This implies that there are C^1 smooth Lipschitz bumps in ℓ_2 so that the cones generated by their sets of gradients have empty interior. On the other hand, we construct C^1-smooth bounded starlike bodies $A \subset \ell_2$, which approximate the unit ball, so that the cones generated by the hyperplanes which are tangent to A have empty interior as well. We also explain why this is the best answer to the above questions that one can expect.

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RÉSUMÉ (*Sur la taille des ensembles de dérivées des fonctions bosses et des hyperplans tangents aux corps étoilés dans l'espace de Hilbert*)

On étudie la taille des ensembles de dérivées des fonctions bosses sur l'espace de Hilbert ℓ_2, ainsi que celle de l'ensemble des hyperplans tangents à un corps étoilé dans ℓ_2. On trouve que ces ensembles peuvent être assez petits. D'un côté, la norme de l'espace de Hilbert peut s'approximer uniformément par des fonctions de classe C^1 et lipschitziennes ψ telles que les cônes générés par les images des dérivées $\psi'(\ell_2)$ sont d'intérieur vide. Cela entraîne l'existence de fonctions de classe C^1 et lipschitziennes dont les cônes générés par les images des dérivées sont d'intérieur vide. On construit d'autre part des corps étoilés bornés lisses de classe C^1 et lipschitziens dont les cônes générés par leurs hyperplans tangents sont d'intérieur vide. On montre aussi pourquoi ces résultats constituent la meilleure réponse à ces questions que l'on puisse espérer.

1. Introduction

Smooth bump functions and starlike bodies are objects that arise naturally in non-linear functional analysis, and therefore their geometrical properties are worth studying. However, very natural questions about tangent hyperplanes to such objects have remained unasked or unanswered, even in the Hilbert space, until very recently.

For instance, if $b : X \to \mathbb{R}$ is a smooth bump on a Banach space X (that is, a smooth function with a bounded support, not identically zero), how many tangent hyperplanes does its graph have? In other words, if we denote the cone generated by its set of gradients by

$$\mathcal{C}(b) = \big\{\lambda b'(x) : x \in X,\ \lambda \geq 0\big\},$$

what is the (topological) size of $\mathcal{C}(b)$?

This problem is strongly related to a similar question about the size of the cones of tangent hyperplanes to starlike bodies in X. Namely, if A is a smooth bounded starlike body in X, how many tangent hyperplanes does A have? More precisely, if we denote the cone of hyperplanes which are tangent to A at some point of its boundary ∂A by

$$\mathcal{C}(A) = \big\{x^* \in X : x + \operatorname{Ker} x^* \text{ is tangent to } \partial A \text{ at some point } x \in \partial A\big\},$$

what is the size of $\mathcal{C}(A)$?

Although in this paper we are mainly concerned with the case of the Hilbert space ℓ_2, it may be helpful to make some previous general considerations about these questions.

To begin with, as a consequence of Ekeland's variational principle [4], it is easily seen that if $b : X \to \mathbb{R}$ is a Gâteaux smooth and continuous bump function on a Banach space X then the norm-closure of $b'(X)$ is a neighbourhood of 0 in X^*. If, in addition, X is finite-dimensional, and b is C^1 smooth, then $b'(X)$ is a compact neighbourhood of 0 in X^*, and in particular 0 is an interior point of $b'(X)$.

However, the classical Rolle's theorem is false in a Banach space X whenever there are smooth bumps in X (see [2] and the references included therein), and this fact has some interesting consequences on the question about the minimal size of the cones of gradients $\mathcal{C}(b)$. Indeed, by using the main result of [2], one can construct smooth bump functions whose sets of gradients lack not only the point zero, but any pre-set finite-dimensional linear subspace of the dual space, so that they violate Rolle's theorem in a quite strong way, as we will see in Section 2.

If we restrict the scope of our search to classic Banach spaces, much stronger results are available. On the one hand, if $X = c_0$ the size of $\mathcal{C}(b)$ can be really small. Indeed, as a consequence of P. Hájek's work [6] on smooth functions on c_0 we know that if b is C^1 smooth with a locally uniformly continuous derivative (note that there are bump functions with this property in c_0), then $b'(X)$ is contained in a countable union of compact sets in X^* (and in particular $\mathcal{C}(b)$ has empty interior). On the other hand, if X is non-reflexive and has a Fréchet norm, there are Fréchet smooth bumps b on X so that $\mathcal{C}(b)$ has empty interior, as it was shown in [1].

In the reflexive case, however, the problem is far from being settled. To begin with, the cone $\mathcal{C}(b)$ cannot be very small, since it is going to be a residual subset of the dual X^*. Indeed, as a consequence of Stegall's variational principle (see [9]), for every Banach space X having the Radon-Nikodym Property (RNP) it is not difficult to see that $\mathcal{C}(b)$ is a residual set in X^*. Thus, for infinite-dimensional Banach spaces X enjoying RNP (such is the case of reflexive ones and, of course, ℓ_2) one can hardly expect a better answer to the question about the minimal size of the cones of gradients of smooth bumps than the following one: there are smooth bumps b on X such that the cones $\mathcal{C}(b)$ have empty interior in X^*.

In the same way, if A is a bounded starlike body in a RNP Banach space then the cone $\mathcal{C}(A)$ of tangent hyperplanes to A contains a subset of second Baire category in X^*, so the best result one could get about the smallest possible size of the cone of tangent hyperplanes to a starlike body in ℓ_2 is that there exist smooth bounded starlike bodies A in ℓ_2 so that $\mathcal{C}(A)$ have empty interior.

In [1] a study was initiated on the topological size of the set of gradients of smooth functions and starlike bodies. Among other results it was proved that an infinite-dimensional Banach space has a C^1 smooth Lipschitz bump function if and only if there exists another C^1 smooth Lipschitz bump function b on X with the property that $b'(X)$ contains the unit ball of the dual X^* and, in particular, $\mathcal{C}(b) = X^*$. It was also established that James' theorem fails for starlike bodies, in the following senses. First, for every Banach space X with a separable dual X^*, there exists a C^1 smooth Lipschitz and bounded starlike body A_1 so that $\mathcal{C}(A_1) = X^*$; in particular we see that there is no upper bound on the size of the cone $\mathcal{C}(A)$, even though X is nonreflexive, and therefore the difficult part of James' theorem is false for starlike bodies. Second, there exists a C^1 smooth Lipschitz and bounded starlike body A_2 in ℓ_2 so that $\mathcal{C}(A_2) \neq \ell_2$, and in particular the "easy" part of James' theorem is false too for starlike bodies.

While the first of these results fully answers the question about the maximal size of the cone $\mathcal{C}(A)$, the second one is not so conclusive, and the natural question as to how small $\mathcal{C}(A)$ can be remained open.

Here, in the case of the Hilbert space $X = \ell_2$, we provide full answers to the questions on the smallest possible size of the cones $\mathcal{C}(A)$ and $\mathcal{C}(b)$, for a smooth bounded starlike body A in X and a smooth bump function b on X . In Sections 2 and 3 we construct C^1 smooth bumps b and C^1 smooth starlike bodies A in ℓ_2 so that the cones of gradients $\mathcal{C}(b)$ and $\mathcal{C}(A)$ have empty interior. Moreover, these strange objects can be made to uniformly approximate the norm and the unit ball of ℓ_2 respectively.

2. How small can the set of gradients of a bump be?

As said above, the question as to how small the cone of gradients of a bump can be is tightly related to the failure of Rolle's theorem in infinite-dimensional Banach spaces. We begin by showing how one can use the main result of [2] to construct smooth bump functions whose sets of gradients lack not only the point zero, but any pre-set finite-dimensional linear subspace of the dual space, so that they violate Rolle's theorem in a quite strong manner.

Theorem 2.1. — *Let X be an infinite-dimensional Banach space and W a finite-dimensional subspace of X^*. The following statements are equivalent.*

1) *X has a C^p smooth (Lipschitz) bump function.*

2) *X has a C^p smooth (Lipschitz) bump function f so that $\mathcal{C}(f) \cap W = \{0\}$ and, moreover,*

$$\big\{f'(x) : x \in \operatorname{int}(\operatorname{supp}(f))\big\} \cap W = \varnothing.$$

Proof. — We only need to prove that 1) implies 2). We can write $X = Y \oplus Z$, where $Y = \bigcap_{w^* \in W} \ker w^*$ and $\dim Z = \dim W$ is finite. Let us pick a C^p smooth (Lipschitz) bump function $\varphi : Y \to \mathbb{R}$ such that $\varphi'(y) = 0$ if and only if $y \notin \operatorname{int}(\operatorname{supp}(\varphi))$ (the existence of such a bump φ is guaranteed by Theorem 1.1 in [2]). Let θ be a C^∞ smooth Lipschitz bump function on Z so that $\theta'(z) = 0$ whenever $\theta(z) = 0$. Then the function $f : X = Y \oplus Z \to \mathbb{R}$ defined by $f(y,z) = \varphi(y)\theta(z)$ is a C^p smooth (Lipschitz) bump which satisfies $\{f'(x) : x \in \operatorname{int}(\operatorname{supp}(f))\} \cap W = \varnothing$. Indeed, if $(y,z) \in Y \oplus Z$ we have

$$f'(y,z) = \big(\theta(z)\varphi'(y), \varphi(y)\theta'(z)\big) \in X^* = Y^* \oplus Z^* = Y^* \oplus W.$$

If $(y,z) \in \operatorname{int}(\operatorname{supp}(f))$, then $\theta(z)\varphi'(y) \neq 0$, and hence $f'(y,z) \notin W$ and $C(f) \cap W = \{0\}$. $\square$

The following theorem and its corollary are the main results of this section. This theorem is also the keystone for the construction of a smooth bounded starlike body whose cone of tangent hyperplanes has empty interior (see the next section).

THEOREM 2.2. — *Let $\|\cdot\|$ denote the usual hilbertian norm of ℓ_2. There are C^1 functions $f_\varepsilon : \ell_2 \to (0,\infty)$, $0 < \varepsilon < 1$, which are Lipschitz on bounded sets and have Lipschitz derivatives, so that:*

1) $\lim_{\varepsilon\to 0} f_\varepsilon(x) = \|x\|^2$ *uniformly on ℓ_2;*

2) $\lim_{\varepsilon\to 0} f'_\varepsilon(x) = 2x$ *uniformly on ℓ_2 (that is, the derivatives of the f_ε uniformly approximate the derivative of the squared norm of ℓ_2); and*

3) *the cones $\mathcal{C}(f_\varepsilon)$ generated by the sets of gradients of the f_ε have empty interior, and $f'_\varepsilon(x) \neq 0$ for all $x \in \ell_2$, $0 < \varepsilon < 1$.*

Moreover, the functions $\psi_\varepsilon = (f_\varepsilon)^{\frac{1}{2}}$ are C^1 smooth and Lipschitz, with Lipschitz derivatives. Note, in particular, that $\lim_{\varepsilon\to 0} \psi_\varepsilon = \|\cdot\|$ uniformly on ℓ_2, the cones of gradients $\mathcal{C}(\psi_\varepsilon)$ have empty interior, and $\psi'_\varepsilon(x) \neq 0$ for all $x \in \ell_2$. Besides, for every $r > 0$, the derivatives ψ'_ε approximate the derivative of the norm uniformly on the set $\{x \in \ell_2 : \|x\| \geq r\}$ as ε goes to 0.

COROLLARY 2.3. — *There is a C^1 Lipschitz bump function b on ℓ_2 (with Lipschitz derivative) satisfying that the cone $\mathcal{C}(b)$ generated by its set of gradients has empty interior, and $b'(x) \neq 0$ for every x in the interior of its support.*

Proofs of Theorem 2.2 and Corollary 2.3

We will make use of the following restatement of a striking result due to S.A. Shkarin (see [10]).

THEOREM 2.4 (Shkarin). — *There is a C^∞ diffeomorphism φ from ℓ_2 onto $\ell_2 \setminus \{0\}$ such that all the derivatives $\varphi^{(n)}$ are uniformly continuous on ℓ_2, and $\varphi(x) = x$ for $\|x\| \geq 1$.*

Let us consider, for $0 < \varepsilon < 1$, the diffeomorphism $\varphi_\varepsilon : \ell_2 \to \ell_2 \setminus \{0\}$, $\varphi_\varepsilon(x) = \varepsilon\varphi(x/\varepsilon)$, and the function $U \equiv U_\varepsilon : \ell_2 \to \mathbb{R}$ defined by $U(x) = \varepsilon^2 + \|\varphi_\varepsilon(x)\|^2$. Then U satisfies the following properties:

(i) U is C^∞ smooth;
(ii) $\|x\|^2 \leq U(x) \leq 2\varepsilon^2 + \|x\|^2$ and $\varepsilon^2 \leq U(x)$, for every $x \in \ell_2$;
(iii) $U(x) = \varepsilon^2 + \|x\|^2$, for every $x \in \ell_2$, $\|x\| \geq \varepsilon$;
(iv) $U'(x) \neq 0$ for every $x \in \ell_2$;
(v) U is Lipschitz in bounded sets and U' is Lipschitz.