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HYPERIDEAL POLYHEDRA IN HYPERBOLIC 3-SPACE

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ABSTRACT. — A hyperideal polyhedron is a non-compact polyhedron in the hyperbolic 3-space $\mathbb{H}^3$ which, in the projective model for $\mathbb{H}^3 \subset \mathbb{RP}^3$, is just the intersection of $\mathbb{H}^3$ with a projective polyhedron whose vertices are all outside $\mathbb{H}^3$ and whose edges all meet $\mathbb{H}^3$. We classify hyperideal polyhedra, up to isometries of $\mathbb{H}^3$, in terms of their combinatorial type and of their dihedral angles.

RÉSUMÉ (*Polyèdres hyperidéaux de l'espace hyperbolique de dimension* 3)
Un polyèdre hyperidéal est un polyèdre non-compact de l'espace hyperbolique $\mathbb{H}^3$ de dimension 3 qui, dans le modèle projectif pour $\mathbb{H}^3 \subset \mathbb{RP}^3$, est simplement l'intersection de $\mathbb{H}^3$ avec un polyèdre projectif dont les sommets sont tous en dehors de $\mathbb{H}^3$ et dont toutes les arêtes rencontrent $\mathbb{H}^3$. Nous classifions ces polyèdres hyperidéaux, à isométrie de $\mathbb{H}^3$ près, en fonction de leur type combinatoire et de leurs angles diédraux.

Consider a compact convex polyhedron P, intersection of finitely many half-spaces in one of the three n-dimensional homogeneous spaces, namely the euclidean space $\mathbb{E}^n$, the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$. The boundary of P inherits a natural cell decomposition, coming from the faces of the polyhedron.

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Along each $(n-2)$-face e, we can measure the internal dihedral angle $\alpha_e \in \,]0,\pi[$ between the two $(n-1)$-faces meeting along e. A natural question then arises: If we are given an $(n-1)$-dimensional cell complex X with a weight $\alpha_e \in \,]0,\pi[$ attached to each $(n-2)$-dimensional cell e, is there a convex polyhedron P in $\mathbb{E}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$ whose boundary has the combinatorial structure of this cell complex X, and such that α_e is the dihedral angle of P along the face e?

An explicit computation provides a full answer in the simplest case where X is the boundary of the n-simplex. The solution involves the signatures of various minors of the symmetric $n \times n$-matrix whose ij-entry is $+1$ if $i = j$ and is $-\cos\alpha_{e_{ij}}$, where e_{ij} is the edge joining the i-th vertex to the j-th vertex, if $i \neq j$; see [6], [17]. In particular, the answer is expressed in terms of the signs of polynomials in $\cos\alpha_{e_{ij}}$. Since this condition on the angles α_e is not that easy to handle, one can expect the general case to be quite intractable, and this indeed seems to be the case.

In general, the main technical difficulty is to control the combinatorics as one deforms the polyhedron P. A typical problem occurs when a p-dimensional face becomes $(p-1)$-dimensional, for instance when two vertices collide so that a 1-dimensional face shrinks to one point.

One way to bypass this technical difficulty is to impose additional restrictions which will prevent such vertex collisions and face collapses. For instance, one can require that all dihedral angles α_e are acute, namely lie in the interval $]0,\frac{1}{2}\pi]$. In this context, Coxeter [6] proved that every acute angled compact convex polyhedron in the euclidean space $\mathbb{E}^n$ is an orthogonal product of euclidean simplices, possibly lower dimensional; this reduces the problem to the case of euclidean simplices, which we already discussed. Similarly, Coxeter also proved that every acute angled convex polyhedron in the sphere $\mathbb{S}^n$ is a simplex. The situation is more complex in the hyperbolic space $\mathbb{H}^n$ but, when $n = 3$, Andreev was able to classify all acute angled compact convex polyhedra in $\mathbb{H}^3$ in terms of their combinatorics and their dihedral angles [2].

In hyperbolic space, another approach to prevent vertex collisions is to put these vertices infinitely apart, by considering (non-compact finite volume) *ideal polyhedra*, where all vertices sit on the sphere at infinity $\partial_\infty\mathbb{H}^n$ of $\mathbb{H}^n$. In [11], Rivin classifies all ideal polyhedra in $\mathbb{H}^3$ in terms of their combinatorics and of their dihedral angles. The case of acute angled polyhedra with some vertices at infinity had been earlier considered by Andreev [3].

In this paper, we propose to go one step further by considering polyhedra in $\mathbb{H}^3$ whose vertices are 'beyond infinity', and which we call *hyperideal polyhedra.*

These are best described in Klein's projective model for $\mathbb{H}^3$. Recall that, in this model, $\mathbb{H}^3$ is identified to the open unit ball in $\mathbb{R}^3 \subset \mathbb{RP}^3$, that geodesics of $\mathbb{H}^3$ then correspond to the intersection of straight lines of $\mathbb{R}^3$ with $\mathbb{H}^3$, and that totally geodesic planes in $\mathbb{H}^3$ are the intersection of linear planes with $\mathbb{H}^3$. In this projective model $\mathbb{H}^3 \subset \mathbb{RP}^3$, a *hyperideal polyhedron* is defined as the intersection P of $\mathbb{H}^3$ with a compact convex polyhedron P^{Proj} of $\mathbb{RP}^3$ with the following properties:

1) Every vertex of P^{Proj} is located outside of $\mathbb{H}^3$;

2) Every edge of P^{Proj} meets $\mathbb{H}^3$.

Note that we allow vertices of P^{Proj} to be located on the unit sphere $\partial_\infty \mathbb{H}^3$ bounding $\mathbb{H}^3$, so that hyperideal polyhedra include ideal polyhedra as a special case.

From now on, we will restrict attention to the dimension $n = 3$. Following the standard low-dimensional terminology, we will call *vertex* any 0-dimensional face or cell, an *edge* will be a 1-dimensional face or cell, and we will reserve the word *face* for any 2-dimensional face or cell.

To describe the combinatorics of a hyperideal polyhedron P, it is convenient to consider the *dual graph* Γ of the cell decomposition of ∂P, namely the graph whose vertices correspond to the (2-dimensional) faces of P, and where two vertices v and v' are connected by an edge exactly when the corresponding faces f and f' of P have an edge in common. Note that Γ is also the dual graph of the projective polyhedron P^{Proj} associated to P.

The graph Γ must be *planar*, in the sense that it can be embedded in the sphere $\mathbb{S}^2$. In addition, Γ is 3-*connected* in the sense that it cannot be disconnected or reduced to a single point by removing 0, 1 or 2 vertices and their adjacent edges; this easily follows from the fact that two distinct faces of P^{Proj} can only meet along the empty set, one vertex or one edge. A famous theorem of Steinitz states that a graph is the dual graph of a convex polyhedron in $\mathbb{R}^3$ if and only if it is planar and 3-connected; see [8]. A classical consequence of 3-connectedness is that the embedding of Γ in $\mathbb{S}^2$ is unique up to homeomorphism of $\mathbb{S}^2$; see for instance [9, §32]. In particular, it intrinsically makes sense to talk of the components of $\mathbb{S}^2 - \Gamma$. Note that these components of $\mathbb{S}^2 - \Gamma$ naturally correspond to the vertices of P^{Proj}.

The results are simpler to state if, instead of the internal dihedral angle α_e of the polyhedron P along the edge e, we consider the *external dihedral angle* $\theta_e = \pi - \alpha_e \in \,]0, \pi[$.

THEOREM 1. — *Let Γ be a 3-connected planar graph with a weight $\theta_e \in \,]0, \pi[$ attached to each edge e of Γ. There exists a hyperideal polyhedron P in $\mathbb{H}^3$ with dual graph isomorphic to Γ and with external dihedral angle θ_e along the edge corresponding to the edge e of Γ if and only if the following two conditions are satisfied:*

1) *For every closed curve γ embedded in Γ and passing through the edges e_1, e_2, …, e_n of Γ, $\sum_{i=1}^{n} \theta_{e_i} \geq 2\pi$ with equality possible only if γ is the boundary of a component of $\mathbb{S}^2 - \Gamma$;*

2) *For every arc γ embedded in Γ, passing through the edges e_1, e_2, …, e_n of Γ, joining two distinct vertices v_1 and v_2 which are in the closure of the same*

component A of $\mathbb{S}^2 - \Gamma$ but such that γ is not contained in the boundary of A, $\sum_{i=1}^{n} \theta_{e_i} > \pi$.

In addition, for the projective polyhedron P^{Proj} associated to P, a vertex of P^{Proj} is located on the sphere at infinity $\partial_\infty \mathbb{H}^3$ if and only if equality holds in Condition 1 for the boundary of the corresponding component of $\mathbb{S}^2 - \Gamma$.

Note that Theorem 1 generalizes Rivin's existence result for ideal polyhedra [11].

THEOREM 2. — *The hyperideal polyhedron P in Theorem 1 is unique up to isometry of $\mathbb{H}^3$.*

Theorem 2 was proved by Rivin [10], [11] for ideal polyhedra, and by Rivin and Hodgson [12] (if we use the truncated polyhedra discussed in §1) for the other extreme, namely for hyperideal polyhedra with no vertex on the sphere at infinity. Even in these cases, one could argue that our proof is a little simpler, as it is based on relatively simple infinitesimal lemmas followed by covering space argument, as opposed to the more delicate global argument of Lemma 4.11 of [12]. However, the main point of Theorem 2 is that it is a key ingredient for the proof of Theorem 1, justifying once again the heuristic principle that "uniqueness implies existence". Theorem 2 is the reason why we introduced Condition 2 in the definition of hyperideal polyhedra, as it fails for general polyhedra without vertices in $\mathbb{H}^3$.

Our proof of Theorems 1 and 2 is based on the continuity method pioneered by Aleksandrov [1] and further exploited in [2] and [12]. We first use an implicit function theorem, proved through a variation of Cauchy's celebrated rigidity theorem for euclidean polyhedra [5], to show that a hyperideal polyhedron is locally determined by its combinatorial type and its dihedral angles. We then go from local to global by a covering argument.

Although the generalization of the results of [11] from ideal polyhedra to hyperideal polyhedra is of interest by itself, the real motivation for this work was to provide a proof of the classification of ideal polyhedra which locally controls the combinatorics of the polyhedra involved. Ideal polyhedra play an important role in 3-dimensional geometry, as they can be used as building blocks to construct hyperbolic 3-manifolds through the use of ideal triangulations, possibly not locally finite. To study deformations of a hyperbolic metric on a 3-manifold, it is therefore useful to have a good classification of the deformations of ideal polyhedra within a given combinatorial type. Unfortunately, Rivin's argument in [11] is indirect. He first uses the classification of compact hyperbolic polyhedra by their dual polyhedra [12], where one completely looses control of the combinatorics, and he extends it to ideal polyhedra by passing to the limit as the vertices go to infinity; he then observes that for ideal polyhedra the dual polyhedron does determine the combinatorics of the ideal polyhedron.

In this regard, the local characterization of hyperideal polyhedra by their dihedral angles provided by our Theorem 11, which is already the key technical step in this paper, may be its most useful result for applications.

The paper [10] provides a different approach to a local control of ideal polyhedra through their combinatorics and dihedral angles. The reader is also referred to [13], [14] for the consideration of other rigidity properties of polyhedra in hyperbolic 3-space.

It may also be of interest that Theorems 1 and 2 can be translated into purely euclidean (or at least projective) statements. Indeed, they provide a classification of hyperideal projective polyhedra P^{Proj} modulo the action of the group $\mathrm{PO}(3,1)$ consisting of those projective transformations of $\mathbb{RP}^3$ that respect the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$. This is particularly remarkable when one notices that, for an edge e of $P = \mathbb{H}^3 \cap P^{\mathrm{Proj}}$, the hyperbolic dihedral angle θ_e of P is equal to the euclidean angle between the two circles $\Pi \cap \mathbb{S}^2$ and $\Pi' \cap \mathbb{S}^2$ at their intersection points, where Π and Π' are the two euclidean planes respectively containing the two faces of P that meet along e. By duality, Theorems 1 and 2 also classify convex projective polyhedra whose faces all meet the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$ but whose edges are all disjoint from the closed ball $\mathbb{H}^3 \cup \mathbb{S}^2$, modulo the action of $\mathrm{PO}(3,1)$.

Theorems 1 and 2 for the somewhat simpler case of *strictly ideal polyhedra*, where all vertices of P^{Proj} are outside of the closure of $\mathbb{H}^3$, appeared in [4].

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1. Hyperideal polyhedra

We first recall a few basic facts about the projective model for $\mathbb{H}^3$ (see for instance [16]).

Consider the symmetric bilinear form

$$B\big((X_0,X_1,X_2,X_3),(Y_0,Y_1,Y_2,Y_3)\big) = -X_0Y_0 + X_1Y_1 + X_2Y_2 + X_3Y_3$$

on $\mathbb{R}^4$. In the projective space $\mathbb{RP}^3$, we consider the image $\mathbb{H}^3$ of the set of those $X \in \mathbb{R}^4$ with $B(X,X) < 0$. For the standard embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$, defined by associating to $(x_1,x_2,x_3) \in \mathbb{R}^3$ the point of $\mathbb{RP}^3$ with homogeneous coordinates $(1,x_1,x_2,x_3)$, the subset $\mathbb{H}^3$ just corresponds to the open unit ball in $\mathbb{R}^3$.

The projection $\mathbb{R}^4 \to \mathbb{RP}^3$ induces a diffeomorphism between $\mathbb{H}^3$ and the set H of those $X = (X_0,X_1,X_2,X_3) \in \mathbb{R}^4$ with $B(X,X) = -1$ and $X_0 > 0$. The tangent space T_XH of H at X is equal to the B-orthogonal of X and,