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FOCUSING OF A PULSE WITH ARBITRARY PHASE SHIFT FOR A NONLINEAR WAVE EQUATION

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Abstract. — We consider a system of two linear conservative wave equations, with a nonlinear coupling, in space dimension three. Spherical pulse like initial data cause focusing at the origin in the limit of short wavelength. Because the equations are conservative, the caustic crossing is not trivial, and we analyze it for particular initial data. It turns out that the phase shift between the incoming wave (before the focus) and the outgoing wave (past the focus) behaves like $\ln \varepsilon$, where ε stands for the wavelength.

Résumé (*Focalisation d'impulsion et déphasage arbitraire pour une équation des ondes non-linéaire*)

Nous considérons un système de deux équations des ondes linéaires conservatives, couplées non-linéairement, en dimension trois d'espace. Pour des données initiales radiales de type impulsions courtes, les solutions focalisent à l'origine lorsque la longueur d'onde tend vers zéro. Le caractère conservatif de l'équation fait que la traversée de la caustique n'est pas triviale : nous l'analysons pour des données initiales particulières. Il ressort que le déphasage entre l'onde entrante (avant focalisation) et l'onde sortante (après focalisation) se comporte en $\ln \varepsilon$, où ε représente la longueur d'onde.

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1. Introduction

1.1. Motivation. — Informations on rapidly oscillating solutions to partial differential equations can be provided by WKB approximations, whose first rigorous justification goes back to Lax [14] (see also [12] for a survey of more recent results). This approach yields good results as long as the solution of the eikonal equation remains smooth, that is, before caustics are formed. The influence of such a singular locus on the behavior of solutions to *linear* partial differential equations was explained by Ludwig [15], and Duistermaat [9]; the caustic crossing is mainly described by the Maslov index.

For nonlinear equations, no global theory is available. Formal computations on conservation laws performed in [10] suggest the existence of two distinct notions of critical indexes; a critical index to describe the solution away from caustics, and another one to analyze the solution near caustics. Rigorous proofs for results similar to those stated in [10] are given in [13], [2], and in the more recent articles by Carles and Rauch in the case of pulse-like data (as opposed to wave trains, see *e.g.* [1], [7]), as we now recall.

Consider the initial value problem,

$$\begin{cases}(\partial_t^2-\Delta)u^\varepsilon + a\varepsilon^{p-2}|\partial_t u^\varepsilon|^{p-1}\partial_t u^\varepsilon = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^3,\\ u^\varepsilon{}_{|t=0} = \varepsilon U_0\Big(r, \dfrac{r-r_0}{\varepsilon}\Big), \quad \partial_t u^\varepsilon{}_{|t=0} = U_1\Big(r,\dfrac{r-r_0}{\varepsilon}\Big),\end{cases} \tag{1.1}$$

where $p \geq 2$, $r = |x|$ and $r_0 > 0$. The parameter ε lies in $]0,1]$, and we want to analyze the asymptotics of $\partial_t u^\varepsilon$ in L^∞ as ε goes to zero. We assume that the functions U_0 and U_1 are infinitely differentiable, bounded, and compactly supported in $r > 0$. The last assumption implies that the initial data are pulse like in the limit $\varepsilon \to 0$. The spherical symmetry of the initial data causes focusing at the origin at time $t = r_0$.

The balance between the power of ε (ε^{p-2}) and the power of the nonlinearity ($|\partial_t u^\varepsilon|^{p-1}\partial_t u^\varepsilon$) corresponds to the critical notion of "nonlinear caustic", as named in [10]; this means that nonlinear effects occur at leading order near the focus $(t,x) = (r_0, 0)$, whereas it would not be so if ε^{p-2} was replaced by ε^δ with $\delta > p-2$ (see [7] for the case $\delta = 0$, $1 < p < 2$).

In [5], [6], the following distinctions were proved in the case $a > 0$, that is when the equation is dissipative (see also [4], [8]).

- If $p > 2$, the solution u^ε passes through the focus, and the caustic crossing is described by a (short range) scattering operator, associated to Eq. (1.1) with $\varepsilon = 1$ (see [6]).
- If $p = 2$, then the pulses are absorbed before reaching the focus ([5], see also [11], [13]).

The case $p > 2$, $a \in \mathbb{C}$ is also considered in [4], for small data U_j, with the same conclusion as in the case $a > 0$. It is described by a scattering operator, and the analysis suggests that the equivalent problem for $a > 0$, $p = 2$ leads to a long range scattering operator. The second point would therefore mean that for a dissipative equation, the image of a long range scattering operator may be reduced to the zero function. On the other hand if a is a pure imaginary, then Eq. (1.1) is conservative, therefore the pulses are not absorbed, and the underlying long range scattering operator should not be trivial. We therefore consider in the present article the case where a is a pure imaginary, and $p = 2$.

In [3], the cubic nonlinear Schrödinger equation is analyzed is one space dimension. A semi-classical analysis shows that when suitable initial data are considered, then the solution focuses at one point, and the caustic crossing is described by a long range scattering operator, which gives rise to a "random" phase shift past the focus, inasmuch as it depends on ε (logarithmically). The nonlinear Schrödinger equation which was considered is conservative, but one could argue that the geometry associated to this problem is not natural. This is why we consider here the wave equation, with the idea of underscoring the corresponding phenomenon of arbitrary phase shift (see Th. 1.1 below).

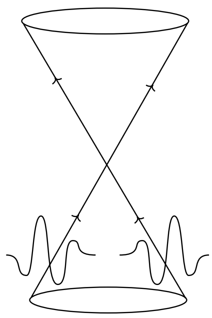

FIGURE 1. Focusing of pulses in the case of radially symmetric initial data for the wave equation.

1.2. Reduction of the problem. — It turns out that the initial value problem (1.1) with $a \in i\mathbb{R}$ and $p = 2$ is technically quite difficult to analyze, with an asymptotic description of the solution in mind. We therefore consider the simplified problem,

$$\begin{cases} (\partial_t^2 - \Delta)\mathbf{u}^\varepsilon = 0, \\ (\partial_t^2 - \Delta)u^\varepsilon - 4i|\partial_t \mathbf{u}^\varepsilon|\partial_t u^\varepsilon = 0. \end{cases} \tag{1.2}$$

This is a system of two linear equations, with a nonlinear coupling that corresponds to a semi-implicit scheme that preserves the conservation of the energy, in view of a numerical treatment for instance. We picked $a = -4i$ for simpler notations in the sequel.

We now proceed to the same reduction as in [7] and [4]. Since the initial data are spherical, so is the solution so, with the usual abuse of notation,

$$u^\varepsilon(t,x) = u^\varepsilon\big(t,|x|\big), \quad u^\varepsilon\big(t,|x|\big) \in C^\infty_{\mathrm{even\,in}\,r}(\mathbb{R}_t \times \mathbb{R}_r).$$

Introduce $v^\varepsilon := (v_-^\varepsilon, v_+^\varepsilon)$ where

$$\tilde{u}^\varepsilon(t,r) := r u^\varepsilon(t,r), \quad v_\mp^\varepsilon := (\partial_t \pm \partial_r)\tilde{u}^\varepsilon. \tag{1.3}$$

Then (1.2) becomes

$$\begin{cases} (\partial_t \pm \partial_r)\mathbf{v}_\pm^\varepsilon = 0, \\ (\partial_t \pm \partial_r)v_\pm^\varepsilon = \frac{i}{r}|\mathbf{v}_-^\varepsilon + \mathbf{v}_+^\varepsilon|(v_-^\varepsilon + v_+^\varepsilon), \quad t \geq 0,\ r > 0, \\ \mathbf{v}_-^\varepsilon + \mathbf{v}^\varepsilon_{+|r=0} = v_-^\varepsilon + v^\varepsilon_{+|r=0} = 0. \end{cases} \tag{1.4}$$

We now turn to the choice of the initial data. As shown in [7], the interaction of the outgoing wave (v_+^ε) and the incoming wave (v_-^ε) is negligible outside the focus, because of the pulse like aspect of the waves (they do not have time to interact), therefore we simplify the notations by imposing $\mathbf{v}^\varepsilon_{+|t=0} = v^\varepsilon_{+|t=0} = 0$. We also choose

$$\mathbf{v}^\varepsilon_{-|t=0} = f\Big(\frac{r - r_0}{\varepsilon}\Big),$$

where $f \in C_0^\infty(\mathbb{R})$. We removed the dependence of the initial data upon slow variables, for it is negligible because of the pulse like aspect. We therefore have explicitly, for $t \geq 0$,

$$\mathbf{v}_-^\varepsilon(t,r) = f\Big(\frac{r + t - r_0}{\varepsilon}\Big), \quad \mathbf{v}_+^\varepsilon(t,r) = -f\Big(\frac{t - r - r_0}{\varepsilon}\Big).$$

The expression of $\mathbf{v}_+^\varepsilon$ shows that on traversing the focus the amplitude of the profile is multiplied by $-1 = \mathrm{e}^{i\mathbf{2}\pi/2}$. This phenomenon is linear: it is the classical Maslov index for a focal point of multiplicity equal to 2 (see *e.g.* [9]).

The choice of $v^\varepsilon_{-|t=0}$ may seem more intricate, but it turns out that it simplifies the computations (at least it makes them feasible) and leads to the phenomenon we want to underscore. We choose

$$v^\varepsilon_{-|t=0} = g\Big(\frac{r - r_0}{\varepsilon}\Big)\mathrm{e}^{i|f((r-r_0)/\varepsilon)|\ln r_0\varepsilon/r}, \tag{1.5}$$

with $g \in C_0^\infty(\mathbb{R})$. The introduction of a logarithmic factor in the phase may seem artificial, just as well as in [3]. Remind that our goal is to describe the caustic crossing thanks to a long range scattering operator: it is classical that this analysis involves phase modification. It will appear later on that our proofs highly rely on this particularity (see Remark 4.2), and it would be interesting to know what happens when this initial phase term is removed. On the other hand, the presence of r_0/r in the logarithmic term is purely cosmetic, to simplify as much as possible the notations in the sequel. It could be removed, essentially because on the support of f, we have $r - r_0 = O(\varepsilon)$.

The reduced problem we will study therefore reads,

$$\begin{cases} (\partial_t \pm \partial_r) v_\pm^\varepsilon = \dfrac{i}{r}\Big| f\Big(\dfrac{r+t-r_0}{\varepsilon}\Big) - f\Big(\dfrac{t-r-r_0}{\varepsilon}\Big)\Big| (v_-^\varepsilon + v_+^\varepsilon), \\ v_-^\varepsilon + v_{+|r=0}^\varepsilon = 0, \\ v_{-|t=0}^\varepsilon = g\Big(\dfrac{r-r_0}{\varepsilon}\Big) \mathrm{e}^{i|f((r-r_0)/\varepsilon)| \ln r_0\varepsilon/r}, \\ v_{+|t=0}^\varepsilon = 0. \end{cases} \tag{1.6}$$

1.3. Statement of the results. — In the rest of this paper, we analyze the reduced functions $v_\pm^\varepsilon$. One could deduce the asymptotics of $\partial_t u^\varepsilon$ in L^∞ thanks to (1.3). The main result of this article is the following.

THEOREM 1.1. — *Let $f, g \in C_0^\infty(\mathbb{R})$, $r_0 > 0$, $\varepsilon > 0$. Then (1.6) has a unique, global solution $(v_-^\varepsilon, v_+^\varepsilon) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)^2$, uniformly bounded for $\varepsilon \in]0,1]$. Moreover, one has the following asymptotics, as ε goes to zero. Let $C > 0$.*

- *If $0 \le t \le r_0 - C\varepsilon$, then*
$$\Big\| v_-^\varepsilon(t,r) - g\Big(\frac{r+t-r_0}{\varepsilon}\Big) \mathrm{e}^{i|f((r+t-r_0)/\varepsilon)| \ln r_0\varepsilon/r} \Big\|_{L_r^\infty} + \big\| v_+^\varepsilon(t,r)\big\|_{L_r^\infty} = O\Big(\frac{\varepsilon}{r_0 - t}\Big).$$
- *There exists $\nu_+ \in L^\infty(\mathbb{R})$ such that for $t \ge r_0 + C\varepsilon$,*
$$\Big\| v_+^\varepsilon(t,r) - \nu_+\Big(\frac{t-r-r_0}{\varepsilon}\Big) \mathrm{e}^{i\theta^\varepsilon(t,r)} \Big\|_{L_r^\infty} + \big\| v_-^\varepsilon(t,r)\big\|_{L_r^\infty} = O\Big(\varepsilon + \frac{\varepsilon}{t-r_0}\Big),$$
where θ^ε is given by
$$\theta^\varepsilon(t,r) = \int_{r_0}^{r/\varepsilon} \frac{1}{\sigma}\Big| f\Big(\frac{t-r-r_0}{\varepsilon} + 2\sigma\Big) - f\Big(\frac{t-r-r_0}{\varepsilon}\Big)\Big| \mathrm{d}\sigma.$$
- *There exists a "caustic profile" $(\mathtt{V}_-, \mathtt{V}_+) \in L^\infty(\mathbb{R} \times \mathbb{R}_+)^2$ such that for $|t - r_0| \le C\varepsilon$ and $r \le C\varepsilon$,*
$$v_\pm^\varepsilon(t,r) = \mathtt{V}_\pm\Big(\frac{t-r_0}{\varepsilon}, \frac{r}{\varepsilon}\Big) + O(\varepsilon).$$

REMARK 1.2. — The constant C in the above statement is arbitrary, its influence is hidden in the remainders. Notice that when $t - r_0 = O(\varepsilon)$, the first two assertions claim nothing more than the uniform boundedness of v_-^ε and v_+^ε.

REMARK 1.3. — We will prove in Sect. 4 that ν_+ is not only bounded, but has algebraic decay, $\nu_+(\lambda) = O(\langle\lambda\rangle^{-1})$, where as usual, $\langle\lambda\rangle = \sqrt{1+\lambda^2}$.