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SCHÉMAS EN GROUPES ET IMMEUBLES DES GROUPES EXCEPTIONNELS SUR UN CORPS LOCAL. PREMIÈRE PARTIE : LE GROUPE G_2

PAR WEE TECK GAN & JIU-KANG YU

RÉSUMÉ. — Nous obtenons une version explicite de la théorie de Bruhat-Tits pour les groupes exceptionnels de type G_2 sur un corps local. Nous décrivons chaque construction concrètement en termes de réseaux : l'immeuble, les appartements, la structure simpliciale, les schémas en groupes associés. Les appendices traitent de l'analogie avec les espaces symétriques réels et des espaces symétriques associés à G_2 réel et complexe.

ABSTRACT (*Group Schemes and Buildings of Exceptional Groups over a Local Field. First Part : the Group G_2*)

We give an explicit Bruhat-Tits theory for the exceptional group of type G_2 over a local field. We describe every construct concretely in terms of lattices: the building, the apartments, the simplicial structure, and the associated group schemes. The appendices discuss analogy with symmetric spaces and the symmetric space of the real or complex G_2.

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WEE TECK GAN, Mathematics Department, Princeton University, Princeton, NJ 08544 (USA)
E-mail : wtgan@math.princeton.edu • *Url :* http://www.math.princeton.edu/~wtgan
JIU-KANG YU, Mathematics Department, University of Maryland, College Park, MD 20742 (USA) • *E-mail :* yu@math.umd.edu • *Url :* http://www.math.umd.edu/~yu

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1. Introduction

The title of this paper is chosen as a tribute to the fundamental contributions of Bruhat and Tits to the structure theory of reductive groups over local fields through their series of papers [3], [4], [5], [7], [6], and is as far as we dare to venture with the French language. In [3], [4], Bruhat and Tits attach to any connected reductive group G over a local field k its building $\mathcal{B}(G)$, which is a polysimplicial complex equipped with an action of $G(k)$. To each point $x \in \mathcal{B}(G)$, they also attach a smooth connected affine group scheme $\underline{G}_x$ over the ring of integers A, with generic fiber G and such that (at least when G is simply-connected) $\underline{G}_x(A)$ is the stabilizer of x in $G(k)$. The description of $\mathcal{B}(G)$ in [3], [4] is given in terms of the notion of valuations of root datum. However, in [5], [7], the building of a classical group G is given a more concrete description in terms of the standard representation V of G: $\mathcal{B}(G)$ is realized geometrically as a set of norms, or equivalently a set of graded lattice chains, on V satisfying certain conditions, and the group schemes $\underline{G}_x$ are realized as stabilizers of these lattice chains in V. Using such a concrete description of $\mathcal{B}(G)$, one can give a lattice-theoretic description of the Moy-Prasad filtration on the parahoric subgroups of classical groups (*cf.* [20] and [19]). In view of such applications, it is useful to extend this concrete description of $\mathcal{B}(G)$ to the case when G is an exceptional group, and the objective of the present paper is to carry out such a programme for the exceptional group of type G_2.

The reader familiar with Bruhat-Tits theory will be disappointed to learn that we will be working over a field which is complete with respect to a discrete valuation. Such a restriction would be considered a sin in [3], [4], but is already present to some extent in [5], [7]. Hence, throughout the paper, A will denote a complete discrete valuation ring, with valuation map ord, field of fractions k, uniformizer π, and perfect residue field A/π of characteristic p. Let G denote a simple algebraic group over k of type G_2; we remind the reader that if the residue field A/π has cohomological dimension ≤ 1 (*e.g.* if A/π is a finite field or is algebraically closed), then such a group is necessarily split [6]. Although all our main results are valid for an arbitrary form of G_2, we will assume that G is the split form of G_2 in most part of the paper. The non-split case, which is very easy, is treated in §12.

The group G can be constructed as the automorphism group of an octonion algebra V over k, and thus has a natural 8-dimensional rational representation. We call V the standard representation of G. Though this representation is not irreducible, it seems more natural to describe the building $\mathcal{B}(G)$ in terms of V, rather than, say, the 7-dimensional submodule of trace zero elements in V.

The octonion algebra V possesses a natural quadratic form which is preserved by G. Hence the representation V gives an embedding $\iota : G \hookrightarrow \mathrm{SO}(V)$. We show in §4 that this gives rise to a canonical embedding $\iota_* : \mathcal{B}(G) \hookrightarrow \mathcal{B}(\mathrm{SO}(V))$. The building $\mathcal{B}(\mathrm{SO}(V))$ has been described explicitly in [7] as the set of maximinorante norms on V (relative to the natural quadratic form on V). Our main results can now be summarized as follows.

(a) The determination of the image of ι_* (Thm. 7.2). The answer is most natural: $\mathcal{B}(G)$ is simply the set of maximinorante norms which are algebra norms for the octonion multiplication. This describes $\mathcal{B}(G)$ as a metric space.

(b) The description of the simplicial complex structure of $\mathcal{B}(G)$, in terms of certain orders in the octonion algebra (Thm. 9.5). Using these orders, we describe the parahoric subgroups of $G(k)$, as well as their associated smooth group schemes over A (Thm. 10.1). We also describe the structure of apartments in $\mathcal{B}(G)$ (Prop. 8.1).

(c) There is a S_3-action on $\mathrm{Spin}(V)$ whose group of fixed points is G. This induces an action of S_3 on $\mathcal{B}(\mathrm{Spin}(V))$. We show that $\mathcal{B}(G)$ is precisely the set of points on $\mathcal{B}(\mathrm{Spin}(V))$ fixed under this action (Cor. 11.4).

The determination of the image of ι_* is an application of a general formalism described in §3 (Thm. 3.5). This formalism is quite useful for identifying the image of a descent map. In addition to (a) and (c), it can be applied to:

(d) The determination of the building of a classical group as a subset of the building of the ambient general linear group (Prop. 4.1). This reproves the results of [7] concerning the buildings of classical groups, at least when the residue characteristic p is not 2.

(e) An explicit description of the building of the split group Spin_8 (Thm. 11.3), together with the action of S_3; this will be needed in the study of the building of a general trialitarian Spin_8.

When $p \neq 2, 3$, the results (c) and (d) also follow from the general results of [13]. The proof here is valid also in residue characteristic 2 or 3, and has the advantage/disadvantage of offering/requiring more information about the arithmetic and geometry underlying the groups involved. The formalism (Thm. 3.5) will also be useful in the study of the buildings of the other exceptional groups.

As is well-known, the reduced Bruhat-Tits building of $G(k)$ is the p-adic analogue of the symmetric space of a reductive real Lie group. In the Appendix §13, we introduce the notion of the extended symmetric space, which is the real analogue of the extended building and which has better functorial properties than the symmetric space. We also prove real analogues of (c) and (d), and more generally the analogue of the main theorem in [13]. Finally, in §14, we prove a real analogue of (a) (Thm. 7.2), which describes the symmetric space of $G_2(\mathbb{R})$ in terms of self-dual norms of an octonion algebra.

2. Generalities on Norms

In this section, let V be a finite dimensional vector space over k. We shall recall some basic notions about norms on V. The material is largely taken from [5], [7], and we include it here for the convenience of the reader and for ease of reference.

A *norm* on V is a function $\alpha : V \to \mathbb{R} \cup \{\infty\}$ satisfying:

– $\alpha(x+y) \geq \inf\{\alpha(x), \alpha(y)\}$, for all $x, y \in V$;
– $\alpha(\lambda x) = \mathrm{ord}(\lambda) + \alpha(x)$, for $\lambda \in k$ and $x \in V$;
– $\alpha(x) = \infty$ if and only if $x = 0$.

A basis $\{x_1, \ldots, x_n\}$ of V is called a *splitting basis* for α if

$$\alpha\Big(\sum_i \lambda_i x_i\Big) = \inf_i \alpha(\lambda_i x_i).$$

Since we are assuming that k is complete with respect to a discrete valuation, every norm α possesses a splitting basis [5, 1.5]. Moreover, if β is another norm on V, there is a common splitting basis for α and β. For each $0 \leq t \leq 1$, there is a norm γ_t which is characterized by the property that any common splitting basis $\{x_1, \ldots, x_n\}$ for α and β is also a splitting basis for γ_t, and

$$\gamma_t(x_i) = t\alpha(x_i) + (1-t)\beta(x_i), \qquad i = 1, \ldots, n.$$

Another way of characterizing γ_t is to say that it is the smallest norm satisfying

$$\gamma_t(x) \geq t\alpha(x) + (1-t)\beta(x), \quad \text{for all } x \in V.$$

We shall denote γ_t by $t\alpha + (1-t)\beta$. This defines an affine structure on the set of norms on V.

The norm α determines a norm α^* on the dual space V^*, which is given by

$$\alpha^*(\varphi) = \inf_{x \in V}\big(\mathrm{ord}(\varphi(x)) - \alpha(x)\big).$$

More concretely, if $\{x_1, \ldots, x_n\}$ is a splitting basis for α, then α^* is characterized by the requirement that it is split by the dual basis $\{x_1^*, \ldots, x_n^*\}$, and $\alpha^*(x_i^*) = -\alpha(x_i)$. Moreover, we have

$$(t\alpha + (1-t)\beta)^* = t\alpha^* + (1-t)\beta^*, \tag{1}$$

for norms α and β on V.

If W is another finite-dimensional vector space over k, equipped with a norm β, then one can form a norm $\alpha \otimes \beta$ on $V \otimes W$, which is given as follows. Let $\{x_1, \ldots, x_n\}$ be a splitting basis for α. Then any element of $V \otimes W$ can be written in the form $\sum_i x_i \otimes w_i$, and

$$(\alpha \otimes \beta)\Big(\sum_i x_i \otimes w_i\Big) = \inf_i \big(\alpha(x_i) + \beta(w_i)\big).$$

In particular, since every non-zero vector is an element of a splitting basis for any norm, we have

$$(\alpha \otimes \beta)(v \otimes w) = \alpha(v) + \beta(w), \qquad \text{for } v, w \neq 0.$$

Now suppose that V is equipped with a non-degenerate bilinear form f, and thus an isomorphism $V \to V^*$ given by: $x \mapsto f(x, -)$. Via this isomorphism, we can regard α^* as a norm on V, and we say that α is *self-dual* (with respect to the given bilinear form f) if $\alpha = \alpha^*$. By (1), one sees that the set of self-dual norms is a convex subset of the set of all norms, in the sense that $t\alpha + (1-t)\beta$ is self-dual if α and β are.

There is another way of viewing the self-dual norms. Suppose that (q, f) is a pair consisting of a non-degenerate quadratic form q and the associated symmetric bilinear form f, so that

$$f(x, y) = q(x + y) - q(x) - q(y).$$

Following [7], we say that a norm α *minorizes* f if it satisfies

$$\alpha(x) + \alpha(y) \leq \operatorname{ord}\big(f(x, y)\big), \quad \text{for all } x, y \in V.$$

It was shown in [7, Prop. 2.5 (ii)] that α is self-dual with respect to f if and only if it is a maximal element in the set of norms minorizing f. Similarly, say that α minorizes (q, f) if it minorizes f, and satisfies

$$\alpha(x) \leq \frac{1}{2} \cdot \operatorname{ord}(q(x)), \quad \text{for all } x \in V.$$

If α is a maximal element in the set of norms minorizing (q, f), then we say that α is *maximinorante* (suppressing the mention of (q, f)). Note that if the residue characteristic p is not 2, then α minorizes f if and only if it minorizes (q, f), and hence α is self-dual if and only if it is maximinorante. The situation is more complicated when $p = 2$. For example, when $\dim(V)$ is odd, the form f is degenerate if $\operatorname{char}(k) = 2$, so that there is no notion of self-duality; even if $\operatorname{char}(k) = 0$, it is possible to have a norm α which minorizes f but not