Bull. Soc. math. France
131 (3), 2003, p. 373–398

RELATIVE EXACTNESS MODULO A POLYNOMIAL MAP AND ALGEBRAIC $(\mathbb{C}^p,+)$-ACTIONS

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ABSTRACT. — Let $F=(f_1,\dots,f_q)$ be a polynomial dominating map from $\mathbb{C}^n$ to $\mathbb{C}^q$. We study the quotient $\mathcal{T}^1(F)$ of polynomial 1-forms that are exact along the generic fibres of F, by 1-forms of type $\mathrm{d}R+\sum a_i\mathrm{d}f_i$, where $R,a_1,\dots,a_q$ are polynomials. We prove that $\mathcal{T}^1(F)$ is always a torsion $\mathbb{C}[t_1,\dots,t_q]$-module. Then we determine under which conditions on F we have $\mathcal{T}^1(F)=0$. As an application, we study the behaviour of a class of algebraic $(\mathbb{C}^p,+)$-actions on $\mathbb{C}^n$, and determine in particular when these actions are trivial.

RÉSUMÉ (*Exactitude relative modulo une application polynomiale et actions algébriques de* $(\mathbb{C}^p,+)$)

Soit $F=(f_1,\dots,f_q)$ une application polynomiale dominante de $\mathbb{C}^n$ dans $\mathbb{C}^q$. Nous étudions le quotient $\mathcal{T}^1(F)$ des 1-formes polynomiales qui sont exactes le long des fibres génériques de F, par les 1-formes du type $\mathrm{d}R+\sum a_i\mathrm{d}f_i$, où $R,a_1,\dots,a_q$ sont des polynômes. Nous montrons que $\mathcal{T}^1(F)$ est toujours un $\mathbb{C}[t_1,\dots,t_q]$-module de torsion. Nous déterminons ensuite sous quelles conditions sur F ce module est réduit à zéro. En application, nous étudions le comportement d'une classe d'actions algébriques de $(\mathbb{C}^p,+)$ sur $\mathbb{C}^n$, et nous déterminons en particulier quand ces actions sont triviales.

1. Introduction

Let $F=(f_1,\dots,f_q)$ be a dominating polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^q$ with $n>q$. Let $\Omega^k(\mathbb{C}^n)$ be the space of polynomial differential k-forms on $\mathbb{C}^n$.

Texte reçu le 30 janvier 2002, accepté le 9 septembre 2002

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2000 Mathematics Subject Classification. — 14R20, 14R25.

Key words and phrases. — Affine geometry, relative cohomology, invariant theory.

For simplicity, we denote by $\mathbb{C}[F]$ the algebra generated by $f_1, \dots, f_q$, and by $\mathbb{C}(F)$ its fraction field. Our purpose in this paper is to compare two notions of relative exactness modulo F for polynomial 1-forms, and to deduce some consequences on some algebraic groups actions.

The first notion is the *topological relative exactness.* A polynomial 1-form ω is topologically relatively exact (in short: TR-exact) if ω is exact along the generic fibres of F. More precisely this means there exists a Zariski open set U in $\mathbb{C}^q$ such that, for any y in U, the fibre $F^{-1}(y)$ is non-critical and non-empty, and ω has null integral along any loop γ contained in $F^{-1}(y)$.

The second notion is the *algebraic relative exactness.* A polynomial 1-form is algebraically relatively exact (in short: AR-exact) if it is a coboundary of the De Rham relative complex of F (see [13]). Recall this complex is given by the spaces of relative forms

$$\Omega^k_F = \Omega^k(\mathbb{C}^n) / \sum \mathrm{d}f_i \wedge \Omega^{k-1}(\mathbb{C}^n)$$

and the morphisms $\mathrm{d}_F : \Omega^k_F \to \Omega^{k+1}_F$ induced by the exterior derivative.

Definition 1.1. — The module of relative exactness of F is the quotient $\mathcal{T}^1(F)$ of TR-exact 1-forms by AR-exact 1-forms. This is a $\mathbb{C}[F]$-module under the multiplication rule $(P(F), \omega) \mapsto P(F)\omega$.

For holomorphic germs, Malgrange implicitly compared these notions of relative exactness in [13]. He proved that the first relative cohomology group of the germ F is zero if the singular set of F has codimension ≥ 3; in this case, $\mathcal{T}^1(F)$ is reduced to zero. In [2], Berthier and Cerveau studied the relative exactness of holomorphic foliations, and introduced a similar quotient. For polynomials in two variables, Gavrilov [9] proved that $\mathcal{T}^1(f) = 0$ if every fibre of f is connected and reduced. Concerning polynomial maps, we first prove the following result.

Proposition 1.2. — *If F is a dominating map, then $\mathcal{T}^1(F)$ is a torsion $\mathbb{C}[F]$-module.*

In other words, every TR-exact 1-form ω can be written as

$$P(F)\omega = \mathrm{d}R + a_1 \mathrm{d}f_1 + \cdots + a_q \mathrm{d}f_q$$

where $R, a_1, \dots, a_q$ are all polynomials. In [3], the author in collaboration with Alexandru Dimca studied in a comprehensive way the torsion of this module for any polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$. We are going to extend these results in any dimension and determine when $\mathcal{T}^1(F)$ is zero.

Let $F : X \to Y$ be a morphism of algebraic varieties, where Y is equidimensionnal and X may be reducible. A property $\mathcal{P}$ on the fibres of F is *k-generic* if the set of points y in Y whose fibre $F^{-1}(y)$ does not satisfy $\mathcal{P}$ has codimension $> k$ in Y. A *blowing-down* is an irreducible hypersurface V in $\mathbb{C}^n$ such that $F(V)$ has codimension ≥ 2 in $\mathbb{C}^q$. If no such hypersurface exists, we say that F has no blowing-downs. Finally F is non-singular in codimension 1 if its singular set has codimension ≥ 2. It is easy to prove that a non-singular map in codimension 1 has no blowing-downs.

DEFINITION 1.3. — The map F is primitive if its fibres are 0-generically connected and 1-generically non-empty.

Then we show that a polynomial map F is primitive if and only if every polynomial R locally constant along the generic fibres of F can be written as $R = S(F)$, where S is a polynomial. So this definition extends the notion of primitive polynomial (*cf.* [8]).

DEFINITION 1.4. — The map F is quasi-fibered if F is non-singular in codimension 1, its fibres are 1-generically connected and 2-generically non-empty. The map F is weakly quasi-fibered if F has no blowing-downs, its fibres are 1-generically connected and 2-generically non-empty.

THEOREM 1.5. — *Let F be a primitive mapping. If F is a quasi-fibered mapping, then $\mathcal{T}^1(F) = 0$. If F is weakly quasi-fibered, then every TR-exact 1-form ω splits as $\omega = \mathrm{d}R + \omega_0$, where R is a polynomial and $\omega_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q = 0$.*

We apply these results to the study of algebraic $(\mathbb{C}^p,+)$-actions on $\mathbb{C}^n$. Such an action is a regular map $\varphi : \mathbb{C}^p \times \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\varphi\big(u, \varphi(v,x)\big) = \varphi(u+v,x)$$

for all u, v, x. Geometrically speaking, φ is obtained by integrating a system $\mathcal{D} = \{\partial_1, \ldots, \partial_p\}$ of derivations on $\mathbb{C}[x_1,\ldots,x_n]$ that are pairwise commuting and locally nilpotent (see [11]), that is :

$$\forall f \in \mathbb{C}[x_1,\ldots,x_n],\ \exists\, k \in \mathbb{N},\quad \partial_i^k(f) = 0.$$

The ring of invariants $\mathbb{C}[x_1,\ldots,x_n]^{\varphi}$ is the set of polynomials P such that

$$P \circ \varphi = P.$$

Finally φ is *free at the point* x if the orbit of x has dimension p, and *free* if it is free at any point of $\mathbb{C}^n$. The set of points where φ is not free is an algebraic set denoted $\mathcal{NL}(\varphi)$.

DEFINITION 1.6 (condition (H)). — An algebraic $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$ satisfies condition (H) if its ring of invariants is isomorphic to a polynomial ring in $n-p$ variables.

Under this condition, φ is provided with a *quotient map* F (see [16]) defined as follows: If $f_1,\ldots,f_{n-p}$ denote a set of generators of $\mathbb{C}[x_1,\ldots,x_n]^{\varphi}$, then

$$F : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-p},\quad x \longmapsto \big(f_1(x),\ldots,f_{n-p}(x)\big).$$

The generic fibres of F are orbits of the action, but this map need not define a topological quotient: For instance, it does not separate all the orbits. The action φ is *trivial* if it is conjugate by a polynomial automorphism of $\mathbb{C}^n$ to

$$\varphi_0(t_1, \dots, t_p; x_1, \dots, x_n) = (x_1 + t_1, \dots, x_p + t_p, x_{p+1}, \dots, x_n).$$

We are going to search under which conditions the actions satisfying (H) are trivial. According to a result of Rentschler [18], every fix-point free algebraic $(\mathbb{C}, +)$-action on $\mathbb{C}^2$ is trivial. We know [15] that (H) is always satisfied for $(\mathbb{C}, +)$-actions on $\mathbb{C}^3$, but we still do not know if fixed-point free $(\mathbb{C}, +)$-actions on $\mathbb{C}^3$ are trivial (see [11]). In dimension ≥ 4, the works [11], [21] of Nagata and Winkelmann prove that (H) need not be satisfied. For $(\mathbb{C}, +)$-actions satisfying this condition, Deveney and Finston [6] proved that φ is trivial if its quotient map defines a locally trivial $(\mathbb{C}, +)$-fibre bundle on its image.

We are going to see how this last result extends via relative exactness. Let φ be a $(\mathbb{C}^p, +)$-action on $\mathbb{C}^n$ satisfying (H), and consider the following operators:

$$\begin{aligned}
[\mathcal{D}] : (R_1, \dots, R_p) &\longmapsto \det\big(\big(\partial_i(R_j)\big)\big),\\
J : (R_1, \dots, R_p) &\longmapsto \det(\mathrm{d}R_1, \dots, \mathrm{d}R_p, \mathrm{d}f_1, \dots, \mathrm{d}f_{n-p}).
\end{aligned}$$

We say that $[\mathcal{D}]$ (resp. J) vanishes at the point x if, for any polynomials $R_1, \dots, R_p$, we have

$$[\mathcal{D}](R_1, \dots, R_p)(x) = 0 \quad \big(\text{resp. } J(R_1, \dots, R_p)(x) = 0\big).$$

The zeros of $[\mathcal{D}]$ correspond to the points of $\mathcal{NL}(\varphi)$, and the zeros of J are the singular points of F. We generalise Daigle's [4] Jacobian Formula for $(\mathbb{C}, +)$-actions.

Proposition 1.7. — *Let φ be an algebraic $(\mathbb{C}^p, +)$-action on $\mathbb{C}^n$ satisfying condition (H). Then there exists an invariant polynomial E such that*

$$[\mathcal{D}] = E \times J.$$

From a geometric viewpoint, this means that $\mathcal{NL}(\varphi)$ is the union of an invariant hypersurface and of the singular set of F. In particular E is constant if $\operatorname{codim} \mathcal{NL}(\varphi) \geq 2$.

Theorem 1.8. — *Let φ be an algebraic $(\mathbb{C}^p, +)$-action on $\mathbb{C}^n$ satisfying condition (H). If E is constant and F is quasi-fibered, then φ is trivial.*

Therefore the assumption "quasi-fibered" correspond to some regularity in the way that F fibres the orbits. In particular the action is trivial if F defines a topological quotient, *i.e.* if F is smooth surjective and separates the orbits.

COROLLARY 1.9. — *Let φ be an algebraic $(\mathbb{C}, +)$-action on $\mathbb{C}^n$ satisfying condition (H). If F is quasi-fibered, there exists a polynomial P such that φ is conjugate to the action*

$$\varphi'(t; x_1, \dots, x_n) = \big(x_1 + tP(x_2, \dots, x_n), x_2, \dots, x_n\big).$$

COROLLARY 1.10. — *Every algebraic $(\mathbb{C}^{n-1}, +)$-action φ on $\mathbb{C}^n$ such that codim $\mathcal{NL}(\varphi) \geq 2$ is trivial. In particular φ is free.*

We end up with counter-examples illustrating the necessity of the conditions of Theorem 1.8 and its corollaries.

2. Proof of Proposition 1.2

In this section, we establish the first proposition announced in the introduction in two steps. First we describe a TR-exact 1-form ω on every generic fibre of F. Second we "glue" all these descriptions by using the uncountability of complex numbers. To that purpose, we use the following definitions.

For any ideal I, we denote by

$$I\Omega^1(\mathbb{C}^n)$$

the space of polynomial 1-forms with coefficients in I. We introduce the equivalence relation:

$$\omega \simeq \omega' \; [I] \iff \omega - \omega' \in \mathrm{d}\Omega^0(\mathbb{C}^n) + \sum \Omega^0(\mathbb{C}^n)\mathrm{d}f_i + I\Omega^1(\mathbb{C}^n).$$

This equivalence is compatible with the structure of $\mathbb{C}[F]$-module given by the natural multiplication, since $\mathrm{d}\Omega^0(\mathbb{C}^n) + \sum \Omega^0(\mathbb{C}^n)\mathrm{d}f_i$ and $I\Omega^1(\mathbb{C}^n)$ are both $\mathbb{C}[F]$-modules.

LEMMA 2.1. — *Let $F^{-1}(y)$ be a non-empty non-critical fibre of F, where $y = (y_1, \dots, y_q)$. A polynomial 1-form ω is exact on $F^{-1}(y)$ if and only if there exists a polynomial R and some polynomial 1-forms $\eta_1, \dots, \eta_q$ such that*

$$\omega = \mathrm{d}R + \sum_i (f_i - y_i)\eta_i.$$

Proof. — Since ω is exact on $F^{-1}(y)$, it has an holomorphic integral R on this fibre. Since $F^{-1}(y)$ is a smooth affine variety, R is a regular map by Grothendieck's Theorem (see [7, p. 182]). In other words, R is the restriction to $F^{-1}(y)$ of a polynomial, which will also be denoted by R. The $(q+1)$-form $(\omega - \mathrm{d}R) \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q$ vanishes on $F^{-1}(y)$. Since $F^{-1}(y)$ is non-critical, $(f_1 - y_1), \dots, (f_q - y_q)$ define a local system of parametres at any point of $F^{-1}(y)$. So the ideal $((f_1 - y_1), \dots, (f_q - y_q))$ is reduced and we get:

$$(\omega - \mathrm{d}R) \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q \equiv 0 \quad [f_1 - y_1, \dots, f_q - y_q].$$