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ON SYSTEMS OF LINEAR INEQUALITIES

by Masami Fujimori

In celebration of the 70th birthday of Professor Genjiro Fujisaki

Abstract. — We show in detail that the category of general Roth systems or the category of semi-stable systems of linear inequalities of slope zero is a neutral Tannakian category. On the way, we present a new proof of the semi-stability of the tensor product of semi-stable systems. The proof is based on a numerical criterion for a system of linear inequalities to be semi-stable.

Résumé (*Sur certains systèmes d'inégalités linéaires*). — On démontre en détail que la catégorie des systèmes de Roth généraux ou la catégorie des systèmes semi-stables d'inégalités linéaires de pente zéro est une catégorie tannakienne neutre. En chemin, on présente une nouvelle preuve de la semi-stabilité du produit tensoriel de systèmes semi-stables. La preuve découle d'un critère numérique pour qu'un système d'inégalités linéaires soit semi-stable.

Introduction

Let $f_1, \ldots, f_n$ be absolutely linearly independent linear forms in n variables $T_1, \ldots, T_n$ with real algebraic coefficients; $c(1), \ldots, c(n)$ real numbers such that

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$c(1)+\cdots+c(n)=0$; and Q, δ positive real numbers. We are primarily interested in properties of the rational integral solutions to the system of inequalities

$$\bigl|f_i(T_1,\ldots,T_n)\bigr| < Q^{-c(i)-\delta} \quad (Q>1;\ i=1,\ldots,n)$$

when δ is fixed. For example, finiteness of the number of solutions.

Let L be the subfield of the field of real numbers generated by all the coefficients of $f_1,\ldots,f_n$. If we do not seek sharp estimates, then it seems that the nature of the system comes from a descending filtration on the L-vector space $LT_1\oplus\cdots\oplus LT_n$: the family $f_1,\ldots,f_n$ is a basis which induces a basis of the associated graded vector space. The number $c(i)$ is the weight of f_i with respect to the filtration. In fact, one sees easily that finiteness of the number of solutions is independent of choices of such a basis (modulo replacement of δ by a slightly larger exponent). A system with finitely many solutions has been called a general Roth system.

From the viewpoint of filtrations, Faltings and Wüstholz [4] gave a projective geometric picture of the set of (rational) solutions to a (related) system of inequalities. In particular, it is coordinate-free. Faltings [3] has found a resemblance between filtered vector spaces and filtered isocrystals and he called semi-stable (of slope zero) a filtered vector space which gives rise to a general Roth system.

In the present article, we aim at proving that the category of general Roth systems, namely, the category of semi-stable filtered vector spaces of slope zero forms a neutral Tannakian category. It means that the category is equivalent to the category of finite dimensional representations of an affine group scheme over the base field. A key lemma is the one stating that a tensor product of semi-stable filtered vector spaces is again semi-stable. The lemma was used for a second proof of the subspace theorem of Schmidt and Schlickewei by Faltings and Wüstholz [4].

Reversing the order of reasoning, we obtain a new proof of the key lemma which depends on the subspace theorem. *Note that it is not a tautology*, because the original proof of Schmidt and Schlickewei does not require the key lemma. The subspace theorem provides us with a simple numerical criterion (Theorem 2.8) for a filtered vector space to be semi-stable. The key lemma is then a consequence (Corollary 2.9) of the criterion.

Our proof is elementary. The difficult parts are hiding in the subspace theorem and in Minkowski's theorem on the geometry of numbers. In Section 1, we make precise various definitions. The section is expositary. In Section 2, we give the new proof of the key lemma.

NOTATION. — Let $\mathbb{R}$ be the field of real numbers. By $i \gg 0$, we mean a real number i is large enough according to the context. The symbol '$\circ$' indicates composition of morphisms.

1. Category of linear inequalities

Let k be a finite extension field of the rational number field and let L be an algebraic extension field of k.

Definition 1.1 (filtration, slope, and weight [9, p. 82])
For a finite dimensional k-vector space V, a family $F^\bullet$ of L-vector spaces

$$F^iV \subset L \otimes_k V \quad (i \in \mathbb{R})$$

is called an *L-filtration* on V if and only if the conditions

$$\begin{aligned} &F^iV \supset F^jV \quad (i \leq j),\\ &F^{-i}V = L \otimes_k V, \quad F^iV = 0 \quad (i \gg 0) \quad \text{and}\\ &F^iV = \bigcap_{j<i} F^jV \end{aligned}$$

are satisfied. We denote the associated graduation by

$$\mathrm{gr}^w(V, F^\bullet) = F^wV/F^{w+0}V \quad (w \in \mathbb{R}),$$

where

$$F^{w+0}V = \bigcup_{j>w} F^jV.$$

The *slope* M of the filtration is a real number

$$M(V, F^\bullet) = \frac{1}{\dim_k V} \sum_{w\in\mathbb{R}} w \dim_L \mathrm{gr}^w(V, F^\bullet).$$

The slope of the zero-dimensional vector space is not defined. The real numbers w such that $\mathrm{gr}^w(V, F^\bullet) \neq 0$ are called the *weights* of the filtration. We often say V is an *L-filtered k-vector space*, instead of saying that $(V, F^\bullet)$ is a k-vector space with an L-filtration. Similarly, we omit $F^\bullet$ from $M(V, F^\bullet)$ or $\mathrm{gr}^w(V, F^\bullet)$ and abbreviate F^iV to V^i.

Definition 1.2 (subfiltration and quotient filtration). — Let W be a subspace over k of V. The L-filtration on W given by

$$W^i = (L \otimes_k W) \cap V^i \quad (i \in \mathbb{R})$$

is the *sub-L-filtration* on W of V. The L-filtration on V/W defined as

$$(V/W)^i = (V^i + L \otimes_k W)/L \otimes_k W \quad (i \in \mathbb{R})$$

is the *quotient L-filtration* on V/W of V.

Lemma 1.3 (see [4, p. 116]). — *Let W be a proper subspace over k of V, and $F^\bullet$ an L-filtration on V. If we endow W with the subfiltration and V/W with the quotient filtration, then we have*

$$M(V) \dim_k V = M(W) \dim_k W + M(V/W) \dim_k(V/W).$$

Proof. — By definition, the sequences

$$0 \to W^w \longrightarrow V^w \longrightarrow (V/W)^w \to 0,$$

$$0 \to W^{w+0} \longrightarrow V^{w+0} \longrightarrow (V/W)^{w+0} \to 0$$

are both exact. By diagram chase,

$$0 \to \operatorname{gr}^w W \longrightarrow \operatorname{gr}^w V \longrightarrow \operatorname{gr}^w(V/W) \to 0$$

is exact, too. The above equality follows at once. □

Example 1.4. — Let v and u be non-zero elements of a k-vector space V such that

$$V = kv \oplus ku.$$

We attach to V the following filtration:

$$V^i = \begin{cases} L \otimes_k V & \text{for } i \le 0, \\ L(v+u) & \text{for } 0 < i \le 1, \\ 0 & \text{for } i > 1. \end{cases}$$

The subfiltration on $W = kv$ is

$$W^i = \begin{cases} L \otimes_k W & \text{for } i \le 0, \\ 0 & \text{for } i > 0. \end{cases}$$

For the subspace $U = ku$, the quotient filtration on V/U becomes

$$(V/U)^i = \begin{cases} L \otimes_k (V/U) & \text{for } i \le 1, \\ 0 & \text{for } i > 1. \end{cases}$$

This is especially telling that although there is a canonical isomorphism of $W = W/W \cap U$ onto $V/U = (W+U)/U$ as vector spaces, they are not isomorphic as filtered vector spaces. In general, a subquotient filtration is not necessarily defined.

Remark 1.5. — As is easily seen, in the case $U \subset W \subset V$, the *subquotient L-filtration* on W/U of V is well-defined.

Definition 1.6 (filtered homomorphism [4, p. 117]). — For L-filtered k-vector spaces V and W, a *filtered homomorphism* $f\colon V \to W$ is a k-linear map such that

$$f(V^i) \subset W^i \quad (i \in \mathbb{R})$$

when extended over L. It is said to be *strict* if

$$f(V^i) = \bigl[L \otimes_k f(V)\bigr] \cap W^i \quad (i \in \mathbb{R}).$$

The strictness of f means that the k-vector space $V/\operatorname{Ker} f$ with the quotient filtration of V (the *coimage* $\operatorname{Coim} f$ of f) is isomorphic to the k-vector space $f(V)$ with the subfiltration of W (the *image* $\operatorname{Im} f$ of f).

Remark 1.7. — For a subspace W over k of V, the canonical maps $W \to V$ and $V \to V/W$ are strictly filtered with respect to the induced filtrations. A composition of filtered homomorphisms is filtered. In Example 1.4, the canonical map $W \to V/U$ is filtered but not strict. In Remark 1.5, the canonical map $W \to V/U$ is strict.

Lemma 1.8. — *If a filtered homomorphism $f\colon V \to W$ is bijective as a k-linear map, then*

$$M(V) \leq M(W).$$

Moreover, the equality is valid if and only if it is an isomorphim of filtered vector spaces.

Proof. — Induction on the number of weights of V. First note that for the proof, the case $L = k$ is sufficient.

When V has only one weight, the whole claim is almost trivial.

Suppose V has plural weights and w is the largest among them. Let the inclusion map be $g\colon V^w \to V$. We endow $V/\operatorname{Im} g$ and $W/\operatorname{Im} f \circ g$ with the respective quotient filtrations (the *cokernels* $\operatorname{Coker} g$ and $\operatorname{Coker} f \circ g$). The number of weights of $\operatorname{Im} g$ is one, and the number of weights of $\operatorname{Coker} g$ is fewer than the number of weights of V. The inductive assumption yields

$$M(\operatorname{Im} g) \leq M(\operatorname{Im} f \circ g) \quad \text{and} \quad M(\operatorname{Coker} g) \leq M(\operatorname{Coker} f \circ g).$$

From Lemma 1.3 we get the inequality we wanted. Furthermore, when we have $M(V) = M(W)$, the above inequalities must be equalities. By the inductive hypothesis,

$$\operatorname{Im} g \simeq \operatorname{Im} f \circ g \quad \text{and} \quad \operatorname{Coker} g \simeq \operatorname{Coker} f \circ g.$$

In particular,

$$\operatorname{gr}^i \operatorname{Im} g \simeq \operatorname{gr}^i \operatorname{Im} f \circ g \quad \text{and} \quad \operatorname{gr}^i \operatorname{Coker} g \simeq \operatorname{gr}^i \operatorname{Coker} f \circ g \quad (i \in \mathbb{R}).$$

By the third exact sequence in the proof of Lemma 1.3, we obtain

$$\operatorname{gr}^i V \simeq \operatorname{gr}^i W \quad (i \in \mathbb{R}),$$

hence $V \simeq W$. □

Lemma 1.9. — *If a filtered homomorphism $f\colon V \to W$ is injective as a k-linear map, then*

$$M(V) \dim_k V \leq M(W) \dim_k W.$$

Proof. — By definition, the induced morphism

$$V \to \operatorname{Im} f$$

is filtered. Since it is also an isomorphism of k-vector spaces, we obtain by Lemma 1.8 $M(V) \leq M(\operatorname{Im} f)$. From Lemma 1.3, we get the desired inequality. □