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DIFFERENTIAL GALOIS THEORY FOR AN EXPONENTIAL EXTENSION OF $\mathbb{C}((z))$

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ABSTRACT. — In this paper we study the formal differential Galois group of linear differential equations with coefficients in an extension of $\mathbb{C}((z))$ by an exponential of integral. We use results of factorization of differential operators with coefficients in such a field to give explicit generators of the Galois group. We show that we have very similar results to the case of $\mathbb{C}((z))$.

RÉSUMÉ (*Théorie de Galois différentielle*). — On étudie le groupe de Galois différentiel formel d'équations différentielles linéaires dont les coefficients sont dans une extension exponentielle de $\mathbb{C}((z))$. On utilise des résultats de factorisation d'opérateurs différentiels à coefficients dans un tel corps pour expliciter des générateurs du groupe de Galois. On obtient des résultats très similaires au cas du corps $\mathbb{C}((z))$.

1. Introduction

The motivation of this work is to write a local differential Galois theory for linear differential equations with coefficients admitting essential singularities. The aim is to generalize the case of differential equations having germs of meromorphic functions as coefficients. We only treat here the formal case, and work with the field $K = \mathbb{C}((z))$ of formal power series. It is a natural idea to add to this field exponential functions, as these ones are the new functions that appear to build solutions of equations with coefficients in K.

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Let's first fix some notations. We endow the field K with the derivation

$$\delta = -z^2 \frac{\mathrm{d}}{\mathrm{d}z}\cdot$$

We set

$$X = \mathrm{e}^{1/z} \quad \text{and} \quad L = \mathbb{C}\big((z)\big)\big((X)\big).$$

We extend the derivation δ to L by $\delta(X) = X$. We also endow the field K with the z-adic valuation v_z and the field L with the X-adic valuation v_X. We notice that this field is complete with respect to this valuation. We want to study linear differential equations with coefficients in L from the differential Galois theory viewpoint. In particular we want to determine the structure of the differential Galois group. For this we proceed like for the field K and we obtain very similar results. Let's recall the well-known results for the field K.

Definition 1.1. — We call a *universal differential extension* of a differential field k a k-algebra R satisfying the following conditions:

- the derivation defined on k extends to R;
- R is simple (*i.e.* R has no non-trivial differential ideal);
- every homogeneous linear differential equation with coefficients in k has all its solutions in R;
- R is minimal, that is R is generated over k by all the solutions (and their derivatives) of homogeneous linear differential equations with coefficients in k.

For any differential field such an extension exists and is unique up to differential isomorphism. In the case of K, we can explicitly write the universal differential extension. It is given by the following symbols:

$$R = \mathbb{C}\big((z)\big)\big[\{z^m\}_{m\in\mathbb{C}}, \{e(p)\}_{p\in\mathcal{P}}, \ell\big]$$

where $\mathcal{P} = \bigcup_{n\geq 1} z^{-1/n}\mathbb{C}[z^{-1/n}]$, and the following relations:

$$z^{a+b} = z^a z^b, \quad e(p_1+p_2) = e(p_1)e(p_2)$$

and $z^a = z^a \in \mathbb{C}((z))$ for $a \in \mathbb{Z}$. The derivation on R is given by

$$(z^a)' = az^a, \quad \big(e(p)\big)' = pe(p), \quad \ell' = 1.$$

(We also write $'$ the derivation $z\mathrm{d}/\mathrm{d}z$ on $\mathbb{C}((z))$).

We can interpret the preceding symbols as functions, which makes sense on suitable sectors. The symbol z^a can be interpreted as the function $\mathrm{e}^{a\log(z)}$, ℓ as a logarithm function and $e(p)$ as the function $\exp(\int p/z\mathrm{d}z)$.

Once we have the universal differential extension of K we define K-differential automorphisms of R as follows:

- the *formal monodromy* γ is defined by

$$\gamma(z^a) = \mathrm{e}^{2i\pi a}z^a, \quad \gamma(\ell) = \ell + 2i\pi, \quad \gamma\big(e(p)\big) = e\big(\gamma(p)\big);$$

- the *exponential torus* is defined by: for all $h \in \mathrm{Hom}(\mathcal{P}, \mathbb{C}^*)$, σ_h is given by

$$\sigma_h(z^a) = z^a, \quad \sigma_h(\ell) = \ell, \quad \sigma_h\big(e(p)\big) = h(p)e(p).$$

Then the exponential torus and the formal monodromy generate the differential Galois group of the extension $R/\mathbb{C}((z))$ as a pro-algebraic group.

We show in this paper that we have the same kind of result for the field L. We give an explicit description of the universal differential extension of L and we give topological generators of the differential Galois group as a pro-algebraic group.

The first thing to do is then to determine all the solutions of homogeneous linear differential equations with coefficients in L. For this we show that it suffices to solve order 1 homogeneous and non-homogeneous equations with coefficients in the algebraic closure of L. Let's write

$$\widehat{K}_\infty = \bigcup_{n \geq 1} \mathbb{C}\big((z^{1/n})\big).$$

Then $\widehat{K}_\infty$ is the algebraic closure of K. The algebraic closure of L is

$$\widehat{L}_\infty = \bigcup_{m \geq 1, n \geq 1} \mathbb{C}\big((z^{1/n})\big)\big((X^{1/m})\big).$$

(The previous valuations and derivations extend to these fields). We notice that the field of constants of $\widehat{L}_\infty$ as well as the one of all intermediate differential fields is $\mathbb{C}$.

We need the following result of factorization that can be found in [1]:

Theorem. — *Let $P \in \widehat{L}_\infty[\delta]$ a non constant differential operator. Then P can be factored in product of order 1 operators in the ring $\widehat{L}_\infty[\delta]$.*

A right factor immediately gives a formal solution by solving an equation of the type $\delta(y) = ay$. We show by analyzing this factorization that either the factors "commute" in a certain sense, either some order 1 non-homogeneous equations appear.

2. Formal classification of differential equations

We want to determine the solutions of all linear differential equations with coefficients in L, but we are interested only in "new" solutions that are not already in $\widehat{L}_\infty$. Thus we start by classifying order 1 equations over $\widehat{L}_\infty$.

2.1. Order 1 equations

Homogeneous equations. — We want to classify equations of the type $\delta(y) = ay$, with $a \in \widehat{L}_\infty$.

Definition 2.1.1. — *The equations $\delta(y)=ay$ and $\delta(y)=by$, with $a,b\in\widehat{L}_\infty$, are said to be equivalent over $\widehat{L}_\infty$ if there exists $f\in\widehat{L}_\infty\backslash\{0\}$ such that $b-a=\delta(f)/f$. (The solutions of $\delta(y)=ay$ are then the solutions of $\delta(y)=by$ multiplied by f.)*

To classify the order 1 homogeneous equations we have to determine the set

$$\mathcal{L}og=\big\{\delta(f)/f, f\in\widehat{L}_\infty\big\}$$

to study the quotient $\widehat{L}_\infty/\mathcal{L}og$. Some computations show that

$$\mathcal{L}og=\Big\{\lambda+\mu z+\textstyle\sum_{r>0}\alpha_r z^{1+r/n}+\sum_{r>0}\beta_r X^{r/m};\\ n,m\geq 1,\lambda,\mu\in\mathbb{Q},\alpha_r\in\mathbb{C},p\geq 1,\beta_r\in\mathbb{C}\big((z^{1/p})\big)\Big\}.$$

Let M be a $\mathbb{Q}$-vector space such that $M\oplus\mathbb{Q}=\mathbb{C}$. We set:

$$\mathcal{Q}=\bigcup_{m\geq 1,n\geq 1}X^{-1/m}\mathbb{C}\big((z^{1/n})\big)[X^{-1/m}],$$

$$\mathcal{P}=\Big\{\bigcup_{n\geq 1}z^{-1/n}\mathbb{C}[z^{-1/n}]\Big\}\cup\bigcup_{n\geq 1}\Big\{\sum_{r=1}^{n-1}\alpha_r z^{r/n},\ \alpha_r\in\mathbb{C}\Big\}.$$

Then $M\oplus Mz\oplus\mathcal{P}\oplus\mathcal{Q}$ classifies the order 1 homogeneous linear differential equations with coefficients in $\widehat{L}_\infty$. We study each of these sets to define symbolic solutions. We have the following symbols:

$$\{X^m\}_{m\in M},\quad \{z^m\}_{m\in M},\quad \big\{e(p)\big\}_{p\in\mathcal{P}},\quad \big\{g(q)\big\}_{q\in\mathcal{Q}}.$$

The solutions of the equation

$$\delta(y)=(m+\widetilde{m}z+p+q)y$$

are given by

$$y=aX^m z^{\widetilde{m}}e(p)g(g),$$

with $a\in\mathbb{C}$, and those of the equivalent equations by fy, with $f\in\widehat{L}_\infty\backslash\{0\}$.

We observe that these symbols are algebraically independent over $\widehat{L}_\infty$.

As for the symbols defined to solve equations over the field K these symbols can be interpreted as functions, which makes sense on suitable sectors. The symbols X^m and z^m can be interpreted as the function $\mathrm{e}^{m\log(X)}$ and $\mathrm{e}^{m\log(z)}$, the symbols $e(p)$ as the functions $\exp(\int -p/z^2\mathrm{d}z)$ and the symbols $g(q)$ as the functions $\exp(\int -q/z^2\mathrm{d}z)$. The symbols $e(p)$ correspond to the symbols we recalled in the introduction for the field K. (We wrote them $e(p)$ with $p\in\bigcup_{n\geq 1}z^{-1/n}\mathbb{C}[z^{-1/n}]$ with the derivation $z\mathrm{d}/\mathrm{d}z$. The second set comes from this change of derivation. We do not obtain the function $\mathrm{e}^{1/z}$ as it is already in the base field.) The symbols $g(q)$ are exponentials of "second level".

Non homogeneous equations. — We want to classify equations of the type $\delta(y) = a$, with $a \in \widehat{L}_\infty$.

DEFINITION 2.1.2. — The equations $\delta(y) = a$ and $\delta(y) = b$, with $a, b \in \widehat{L}_\infty$, are said to be *equivalent over* $\widehat{L}_\infty$ if there exists $f \in \widehat{L}_\infty$ such that $b - a = \delta(f)$. (The solutions of $\delta(y) = a$ are then the solutions of $\delta(y) = b$ added to f.)

We have to determine the set $\widehat{L}_\infty/\mathcal{D}er$, where

$$\mathcal{D}er = \big\{\delta(f), f \in \widehat{L}_\infty\big\}.$$

Some computations show that the order 1 non-homogeneous equations are classified by

$$\widehat{L}_\infty/\mathcal{D}er = \{\alpha z,\, \alpha \in \mathbb{C}\}.$$

Thus we only have one equation to study. Let's look at $\delta(y) = -z$. We set ℓ the symbol solution of this equation. Then the solutions of all equivalent equations are given by $\ell + g$, $g \in \widehat{L}_\infty$. The set of solutions of all the equations $\delta(y) = f$, with $f \in \widehat{L}_\infty$, is $\{\alpha\ell + g, \alpha \in \mathbb{C}, g \in \widehat{L}_\infty\}$.

The symbol ℓ can be interpreted as a logarithm function and we notice that it is algebraically independent over $\widehat{L}_\infty$ with the other symbols.

2.2. Differential operators and differential modules. — Let's write $\mathcal{D} = \widehat{L}_\infty[\delta]$. Let $A \in \mathrm{Hom}((\widehat{L}_\infty)^n, (\widehat{L}_\infty)^n)$. We define the differential module $\mathcal{M}_A$ associated to the system $\delta(Y) = AY$ by the formulas

$$\delta(e_i) = -\sum_j a_{ij} e_j,$$

where $(e_1, \dots, e_n)$ is the standard basis of $(\widehat{L}_\infty)^n$. To a differential operator $P = \delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0$ we associate a differential system $\delta(Y) = AY$ with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{pmatrix}.$$

The module $\mathcal{M}_A$ is called the differential module associated to the operator P. The modules $\mathcal{M}_A$ and $(\mathcal{D}/\mathcal{D}P)^*$ are isomorphic, where $(\mathcal{D}/\mathcal{D}P)^* = \mathrm{Hom}((\mathcal{D}/\mathcal{D}P), \widehat{L}_\infty)$ is the dual of $(\mathcal{D}/\mathcal{D}P)$.

The two differential systems $\delta(Y) = A_1 Y$ and $\delta(Y) = A_2 Y$ are said *equivalent over* $\widehat{L}_\infty$ if there exists $U \in \mathrm{Gl}((\widehat{L}_\infty)^n, (\widehat{L}_\infty)^n)$ such that

$$A_1 = U^{-1}\delta(U) + U^{-1}A_2 U.$$